In Number Theory, we work within the axiomatic system of natural numbers:

$$\mathbb{N} = \{0, 1, 2, 3, \ldots\} \text{ with operations } +, \times, \text{ and inequality } <,$$

referring as little as possible to the larger number systems that contain $\mathbb{N}$ such as the rationals (fractions) $\mathbb{Q}$ and the reals $\mathbb{R}$.

We will assume all specific arithmetic facts such as $2(26+45) = 142$; $9^2 = (9)(9) = 81$; and $0 < 1 < 2 < \cdots$. We also assume the common laws of algebra, most of which are axioms for a group with the $+$ or $\times$ operation. For any $a, b, c \in \mathbb{N}$, we have:

1.	Closure: $a + b \in \mathbb{N}$	1′.	Closure: $ab \in \mathbb{N}$
2.	Assoiciativity: $(a + b) + c = a + (b + c)$	2′.	Assoiciativity: $(ab)c = a(bc)$
3.	Identity element: $a + 0 = a = 0 + a$	3′.	Identity element: $1a = a = a1$
4.	~~Inverse: $\forall a, \exists b : a + b = 0$~~	4′.	~~Inverse: $\forall a \neq 0, \exists b : ab = 1$~~
5.	Commutativity: $a + b = b + a$	5′.	Commutativity: $ab = ba$

6. Distributivity: $a(b + c) = ab + ac$

Note that additive and multiplicative inverses do not exist *within* $\mathbb{N}$ because $-a$ and $1/a$ are generally not elements of $\mathbb{N}$.

We also have the common properties of inequality. We define $a > b$ to mean $b < a$. For any $a, b, c \in \mathbb{N}$, we have:

7. Trichotomy: Exactly one of the following is true: $a < b$, $a = b$, $a > b$.
8. Compatibility of $<$ with $+$: If $a < b$, then $a + c < b + c$.
9. Compatibility of $<$ with $\times$: If $a < b$ and $c > 0$, then $ac < bc$.

Finally, we state the Induction Property: If $P(n)$ is any proposition depending on the integer n, then:

$$\text{If } P(0) \text{ is true, and } \forall n,\ P(n) \Rightarrow P(n{+}1); \text{ then } P(n) \text{ is true for all } n$$

This is equivalent to the Complete Induction Property:

$$\text{If } P(0) \text{ is true, and } \forall n,\ (P(0), P(1), \ldots, P(n)) \Rightarrow P(n{+}1); \text{ then } P(n) \text{ is true for all } n$$

We give a few examples of elementary propositions proved from these axioms.

PROPOSITION 1. Cancellation: If $a + c = b + c$, then $a = b$.
Proof. We prove the contrapositive. Suppose $a \neq b$. Then by (7), we have either: $a < b$, so that $a + c < b + c$ by (8); or $b < a$ and $b + c < a + c$ by (8). In either case, we have $a + c \neq b + c$ by (7), which is the contrapositive conclusion.

Note: We could have added $-c$ to both sides of $a + c < b + c$, but then we would use negative numbers, which are not in $\mathbb{N}$.

PROPOSITION 2. Zero Property: $0a = 0$.
Proof. We have:

$$\begin{aligned} 0 + 0a &= 0a && \text{by (3)} \\ &= (0 + 0)a && \text{by (3)} \\ &= 0a + 0a && \text{by (6)} \end{aligned}$$

Thus by Cancellation (Prop. 1), we have: $0 = 0a$ as desired.

PROPOSITION 3. Factors of 1: If $1 = ab$, then $a = b = 1$.
Proof. We prove the contrapositive. Suppose $a \neq 1$ or $b \neq 1$. Case 1: If $a = 0$ or $b = 0$ then by the Zero Property, $ab = 0 < 1$. Case 2: If $a > 1$ and $b = 1$, then $ab = a > 1$, and similarly if $a = 1$ and $b > 1$. Case 3: If $a, b > 1$, then $ab > (1)(1) = 1$ by (9) and (3′). In any case, $ab \neq 1$, which is the contrapositive conclusion.