Math 299 **Supplement: Group Examples** **Oct 11, 2013**

Recall that a group is defined by data $(G, *)$, where G is a set and $*$ is a binary operation satisfying the four Group Axioms.

1. Our first example is the modular (clock) arithmetic $(G, *) = (\mathbb{Z}_3, +)$. Here $G = \mathbb{Z}_3 = \{\overline{0}, \overline{1}, \overline{2}\}$, which represent the marks on a 3-hour clock. For any integer n, we define the notation $\overline{n} = \overline{n{+}3} = \overline{n{-}3}$, so that $\overline{3} = \overline{0}$, $\overline{7} = \overline{1}$, $\overline{-1} = \overline{2}$ are all elements of $\mathbb{Z}_3$.

The operation is: $\overline{n} + \overline{m} = \overline{n{+}m}$, for example: $\overline{2} + \overline{2} = \overline{2{+}2} = \overline{4} = \overline{1}$. Thus, for any $\overline{n}, \overline{m} \in \mathbb{Z}_3$, we have $\overline{n} + \overline{m} \in \mathbb{Z}_3$, a closed operation, which is clearly associative. Its identity element is $\overline{0}$, and the inverse of $\overline{n}$ is $\overline{-n} = \overline{3{-}n}$. For example, $\overline{1} + \overline{2} = \overline{3} = \overline{0}$, so $\overline{1}$ and $\overline{2}$ are inverses. Thus, $(\mathbb{Z}_3, +)$ satisfies the Group Axioms.

a. Write the operation table of $(\mathbb{Z}_3, +)$. For a general group $(G, *)$, this is a table where the rows correspond to $a \in G$, the columns to $b \in G$; and in the a-row, b-column, we write the product $a * b$. In our case:

$+$	$\overline{0}$	$\overline{1}$	$\overline{2}$
$\overline{0}$			$\overline{2}$
$\overline{1}$			
$\overline{2}$			

Here the entry in the upper right corner means $\overline{0} + \overline{2} = \overline{2}$.

b. How do the formulas $\overline{0} + \overline{n} = \overline{n}$ and $\overline{n} + \overline{0} = \overline{n}$ appear in the table?

c. How can you find the inverse of each element from the table?

d. Is this group commutative? That is, do we have $\overline{n} + \overline{m} = \overline{m} + \overline{n}$ for all $\overline{n}, \overline{m} \in \mathbb{Z}_3$? How can you tell from the table?

2. The symmetric group of a set S is $(G, *) = (\mathrm{Sym}(S), \circ)$, where:

$$\mathrm{Sym}(S) = \{f : S \to S \text{ bijective functions}\},$$

and the operation $\circ$ is composition of functions: $(f \circ g)(x) = f(g(x))$.

For this problem, let $S = [3] = \{1, 2, 3\}$. We can write functions $f : [3] \to [3]$ with the notation: $f = (f(1), f(2), f(3))$. For example $f = (3, 1, 2)$ means the function with $f(1) = 3$, $f(2) = 1$, $f(3) = 2$, and $g = (2, 1, 3)$ means $g(1) = 2$, $g(2) = 1$, $g(3) = 3$.

To compute a composition like $f \circ g = (3, 1, 2) \circ (2, 1, 3)$, we find: $f(g(1)) = f(2) = 1$, $f(g(2)) = f(1) = 3$, $f(g(3)) = f(3) = 2$, so $f \circ g = (1, 3, 2)$. That is:

$$(3, 1, 2) \circ (2, 1, 3) = (1, 3, 2).$$

With a bit of practice, you can compute this in one step, without writing $f(g(1))$, etc.

a. Write the operation table for this group, computing all the compositions in the 6 rows and columns.

b. Rewrite the table by replacing each triple (a, b, c) with a letter label:

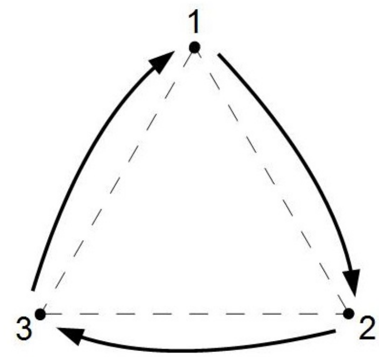
$$e = (1, 2, 3),\ r_1 = (2, 3, 1),\ r_2 = (3, 1, 2),\ s_1 = (1, 3, 2),\ s_2 = (3, 2, 1),\ s_3 = (2, 1, 3).$$

For example, the computation above can be rewritten as: $r_2 \circ s_3 = s_1$.

c. Again, use the table to find the inverse of each element.

d. Is this group commutative? See this immediately from the table.

e. We can picture the group elements by drawing 1,2,3 as vertices of a triangle, and drawing an arrow from each vertex i to the vertex $f(i)$. We can think of f as moving the triangle to itself. For example, $r_1 = (2,3,1)$ has an arrow from 1 to $r_1(1) = 2$, etc., and looks like a $\frac{1}{3}$-rotation:



Draw a similar picture for each element of the group, and describe the resulting motion of the triangle. How can we picture the composition operation in terms of the triangle?