Math 299 **Recitation 3** **Sept 12, 2013**

1. Let the real functions f and g be defined by:

$$f(x) = \sqrt{x} \quad \text{and} \quad g(x) = 1 - x^2.$$

a. Find the largest domain for f so that $f \circ g$ is a real function? Write your answer using proper notation and briefly explain why the domain can't be larger. *Ans:* $f \circ g = \sqrt{1 - x^2}$, domain $D = \{x \in \mathbb{R} \mid -1 \leq x \leq 1\}$.

b. Find a possible codomain for the function f.
Ans: Smallest possible is the image: $\mathrm{Im}(f) = f(D) = \{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$.

2. Consider the function $t : \mathbb{Z} \times \mathbb{Z} \longrightarrow \mathbb{Z} \times \mathbb{Z}$ defined by $t(a, b) = (5a + 2b, 3a + b)$. Write a formula for the inverse function $s : \mathbb{Z} \times \mathbb{Z} \longrightarrow \mathbb{Z} \times \mathbb{Z}$, and verify it is indeed an inverse. Can you conclude that t is injective or surjective?

Ans: To find the inverse, solve for $s(a, b)$ in $t(s(a, b)) = (a, b)$. Denoting $s(a, b) = (c, d)$, we solve for (c, d) in $t(c, d) = (a, b)$; i.e. solve the simultaneous equations:

$$\begin{cases} 5c + 2d &= a \\ 3c + d &= b \end{cases}$$

This gives: $s(a, b) = (-a + 2b, 3a - 5b)$.
Note: This is equivalent to solving the matrix equation:

$$\begin{bmatrix} 5 & 2 \\ 3 & 1 \end{bmatrix} \cdot \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}.$$

The inverse is an integer matrix because det $= -1$.

3. Let $j \leq m$ be natural numbers. Recall that $[m] = \{1, 2, \ldots, m\}$, and $[m]\backslash\{j\}$ means $[m]$ with the element j removed. Consider the bijection $h : [m]\backslash\{j\} \longrightarrow [m-1]$ given by:

$$h(k) = \begin{cases} k & \text{for } k < j \\ k - 1 & \text{for } k > j. \end{cases}$$

Give a formula for the inverse function $g : [m-1] \rightarrow [m]\backslash\{j\}$.

Ans: $g(k) = k$ for $k < j$ and $g(k) = k+1$ for $k \geq j$

4. The natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$ seem to be a smaller set than the integers $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$, though it is not clear what "smaller" means for infinte sets. In fact it is possible to find a bijective function $f : \mathbb{N} \rightarrow \mathbb{Z}$.

Problem: Find such a function f, and write it explicitly, either in cases or as a formula using the floor function $\lfloor \ \rfloor$.

Ans: $f(k) = \frac{1}{2}k$ for k even, $f(k) = -\frac{1}{2}(k{+}1)$ for k odd. Or: $f(k) = (-1)^k \lfloor \frac{k+1}{2} \rfloor$.

5. Consider $A = \{1, 2, 3\}$ and $B = \{w, x, y, z\}$. Justify your answers by citing a previous result or by finding a pattern in the list of all possibilities.

a. How many injective functions are there from A to B? *Ans:* (5)(4)(3) from Supp 9/9, Prop. 3(ii)

b. How many surjective functions are there from B to A?

Ans: The function must take one pair in B to one element in A. First we choose the pair (6 choices), and thereafter we may regard the pair as one unit. Thus 3 units from B are in bijection with A, giving $3! = 6$ choices. Final answer: $(6)(6) = 36$.