Math 299 **Recitation 2** **Sept 5, 2013**

PROBLEM 1.

(a) A coin is flipped twice. Describe the possible outcomes as the elements of a set. What is the cardinality of the set?

(b) Two identical coins are each flipped once. Describe the possible outcomes as the elements of a set. What is the cardinality of the set?

(c) A penny and a dime are each flipped once. Describe the possible outcomes as the elements of a set. What is the cardinality of the set?

PROBLEM 2.

(a) Fix real numbers a, b, c. Use set notation to describe the set of solutions $(x, y) \in \mathbb{R}^2$ to the equation $y = ax^2 + bx + c$.

(b) Let $S = \{a, b, c\}$. Use set notation to describe the set of functions $f : S \to \{0, 1\}$. What is the cardinality of this set?

PROBLEM 3. A line segment in the plane can be described by its endpoints $a = (a_1, a_2)$ and $b = (b_1, b_2)$ in $\mathbb{R}^2$. The set of all line segments in the plane can be described by the set of all possible pairs of endpoints:

$$L = \left\{ \{a, b\} \mid a, b \in \mathbb{R}^2,\ a \neq b \right\}.$$

Note: The elements of L *identify* line segments, but they are not line segments themselves. This is similar to the relationship between a person and his or her name: the name *identifies* the person, but strictly speaking the name is not the person.

(a) Why did we include the requirement $a \neq b$?

(b) Consider the set of ordered pairs:

$$S = \left\{ (a, b) \mid a, b \in \mathbb{R}^2, a \neq b \right\}.$$

Is this the same as L? Can you come up with a geometric interpretation for the elements of the set S? What kind of object is can be specified by a pair of points (a, b)?

(c) Describe the set T of triangles in the plane $\mathbb{R}^2$ in an analogous way.

PROBLEM 4. Here is an example of an interesting problem that we can formulate in terms of combinatorial graphs (see Homework 9/4 handout).

(R) Suppose that we are throwing a party. What is the smallest number of people which guarantees that either some group of m people all know each other, or some group of n people are all strangers to each other?

Given m and n, the answer to this problem is known as the **Ramsey number** $R(m, n)$. Ramsey's Theorem asserts that this is a finite number. For example, suppose there are at least 2 people. If these 2 know each other, then they make a group of 2 who know each other; otherwise, this same group of 2 are strangers. This shows $R(2, 2) = 2$.

(a) Try to find $R(2, 3)$: that is, how many people are needed to guarantee that some group of 3 all know each other, or some group of 2 are strangers? Also, find $R(3, 2)$.

(b) Fix positive integers m and n. Formulate Problem (R) in term of graphs. Use the following questions to help: What do the vertices correspond to in the problem? Suppose there is an edge between two vertices. What does this correspond to? What does it mean if there is no edge between two vertices? What do the numbers m, n and $R(m, n)$ correspond to in terms of the graph?

(c) Can you show $R(m, n) = R(n, m)$?

(d) Try to show $R(3, 3) > 5$: that is, 5 people are not enough to guarantee 3 mutual acquaintances or 3 mutual strangers. Hint: It is enough to find a single acquaintance graph which breaks the guarantee.

(e) Now show $R(3, 3) \leq 6$, meaning 6 people is enough in the case of (d). Together, these imply $R(3, 3) = 6$. (This was a Putnam problem in 1953.)

It is known that $R(4, 4) = 18$, but in general, the numbers $R(m, n)$ are extremely difficult to compute. The prolific mathematician Paul Erdös characterized it thus:

> Suppose aliens invade the earth and threaten to obliterate it in a year's time unless human beings can find the Ramsey number R(5, 5). We could marshal the world's best minds and fastest computers, and within a year we could probably calculate the value. If the aliens demanded the Ramsey number R(6, 6), however, we would have no choice but to launch a preemptive attack.