Math 299 **Recitation 1** **Aug 29, 2013**

PROBLEM 1. A rectangular plot of farmland will be bounded on one side by a river and on the other three sides by a fence. With 800m of fence available, what shape of rectangle will enlcose the largest area?

Solution A.

$$\text{Area} = ab$$

$$2a + b = 800$$

$$b = 800 - 2a$$

$$f(a) = a(800 - 2a)$$

$$f'(a) = 800 - 4a = 0 \implies a = 200, b = 400$$

Solution B. (Improved with explanations)

Denote the length and width of the rectangular plot by a and b. The area is given by:

$$\text{Area} = ab.$$

Without loss of generality [why?], we can assume that the river runs along the width b of the rectangle, and the 800m of fence runs along sides of length a, b, a; hence:

$$2a + b = 800.$$

Solving the above equation for b:

$$b = 800 - 2a.$$

Then the area as a function of the length is:

$$f(a) = a(800 - 2a),$$

where f has domain $[0, 400]$, since a valid length has $a \geq 0$, and $b \geq 0$ whenever $a \leq 400$.

Recalling the method to find the absolute maximum of $f(a)$ in its domain $a \in [0, 400]$:

1. Find the critical points in the domain, where the derivative $f'(a)$ equals zero or is undefined. We have: $f'(a) = 800 - 4a$, which is defined for all a, and $f'(a) = 0$ only for $a = 200$.
2. Evaluate $f(a)$ at the critical points: $f(200) = (200)(400) = 80000$.
3. Evaluate $f(a)$ at the endpoints of the domain: $f(0) = f(400) = 0$.
4. The absolute maximum is the largest value $f(a)$ from steps 2 and 3: that is, $f(200) = 80000$.

Thus, the largest area that can be enclosed is 80,000m^2, achieved when the plot has length $a = 200$m and width $b = 400$m.

PROBLEM 2. Prove the following:

THEOREM: *Among all rectangles with a given fixed area, the one with the smallest perimeter is a square.*

Proof A.

$$xy = A$$

$$y = \frac{A}{x}$$

$$f(x) = 2x + \frac{2A}{x}$$

$$f'(x) = 2 - \tfrac{2A}{x^2} = 0 \implies x = \sqrt{A} \implies y = \sqrt{A} \implies \text{square}$$

Proof B. Consider all rectangles having a fixed area A, and denote the length and width of such a rectangle by x and y. Then $xy = A$, and solving for y gives:

$$y = \frac{A}{x}. \qquad (1)$$

[Does it make a difference whether we solve for x or for y?] The perimeter of the rectangle is $P(x, y) = 2x + 2y$, and substituting from equation (1) expresses the perimeter as a function of the single variable x:

$$f(x) = 2x + \frac{2A}{x},$$

whose domain is $x \in (0, \infty]$, since we must have the length $x > 0$, and this also guarantees $y > 0$.

To find the absolute minimum of f, we first look for critical points in the domain, where $f'(x)$ is zero or undefined. We compute:

$$f'(x) = 2 - \frac{2A}{x^2},$$

which is defined for all x in the domain, and which equals zero only at $x = \sqrt{A}$. Since the interval is open, the endpoint values $f(0)$ and $f(\infty)$ are undefined, but $f(x)$ might still approach (though not achieve) minimum values near the endpoints. However, $\lim_{x\to 0^+} f(x) = \infty$ and $\lim_{x\to\infty} f(x) = \infty$, meaning $f(x)$ gets arbitrarily *large* near these endpoints. Therefore, the minimum value can only be at the critical point $x = \sqrt{A}$.

[Another way to eliminate the endpoints is to show that $f''(x) > 0$ on the whole domain, so that $f(x)$ is concave up, and $f'(x)$ is increasing for all x. Since $f'(x) = 0$ for $x = \sqrt{A}$, this means $f(x)$ must be decreasing for $x < \sqrt{A}$, and increasing for $x > \sqrt{A}$, and hence that $x = \sqrt{A}$ is the unique minimum point.]

We conclude that the rectangle with minimum perimeter has length $x = \sqrt{A}$, which by equation (1) implies the width $y = A/\sqrt{A} = \sqrt{A}$. Since length equals width, the minimum rectangle is a square.