In Supplement 9/9, we defined the choose number, or binomial coefficient, $\binom{n}{k}$ to be the number of possible k-element subsets $S \subset [n]$, where $[n] = \{1, 2, \dots, n\}$. For example, $\binom{4}{2} = 6$ counts the 2-element subsets $S \subseteq \{1, 2, 3, 4\}$, namely: $S = \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}$.

We can put these numbers into an array called Pascal's Triangle (in China, Yang Hui's Triangle; in Iran, Khayyam's Triangle):

$$\begin{array}{ccccccccc} &&&&\binom{0}{0}&&&& \\ &&&\binom{1}{0}&&\binom{1}{1}&&& \\ &&\binom{2}{0}&&\binom{2}{1}&&\binom{2}{2}&& \\ &\binom{3}{0}&&\binom{3}{1}&&\binom{3}{2}&&\binom{3}{3}& \\ \binom{4}{0}&&\binom{4}{1}&&\binom{4}{2}&&\binom{4}{3}&&\binom{4}{4} \\ \vdots&&\vdots&&\vdots&&\vdots&&\vdots \end{array} \quad = \quad \begin{array}{ccccccccc} &&&&1&&&& \\ &&&1&&1&&& \\ &&1&&2&&1&& \\ &1&&3&&3&&1& \\ 1&&4&&6&&4&&1 \\ \vdots&&\vdots&&\vdots&&\vdots&&\vdots \end{array}$$

We can compute the entries by the formula $\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!}$, but there is an easier way. It is a remarkable fact that each entry in the triangle is the sum of the two entries immediately above it (except for the edges $\binom{n}{0} = \binom{n}{n} = 1$). For example, the next row will be:

$$\binom{5}{0} = 1, \quad \binom{5}{1} = \binom{4}{0} + \binom{4}{1} = 5, \quad \binom{5}{2} = \binom{4}{1} + \binom{4}{2} = 10, \quad \binom{5}{3} = \binom{4}{2} + \binom{4}{3} = 10, \ \dots$$

In general, the Recurrence Formula which generates the triangle is:

$$(*) \qquad \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

Problem 1. Use the above recurrence to compute the $\binom{6}{k}$ and $\binom{7}{k}$ rows of the table.

Problem 2. Find the sum of each row: $\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n}$, for $n = 0, 1, \dots, 7$. Guess a general formula for this sum.

Problem 3. Prove your formula from Prob. 2 by asking: what kind of objects are counted by the left side? Then look in Supplement 9/9 and apply one of the propositions.

Next we consider how to prove the Recurrence Formula $(*)$ through the Bijection Principle: we need to write he left and right sides of the formula as the cardinalities (sizes) of some sets $\mathcal{A}$ and $\mathcal{B}$.

- $\binom{n}{k} = |\mathcal{A}|$, where $\mathcal{A}$ is the set of all k-element subsets of $[n]$.
- $\binom{n-1}{k-1} = |\mathcal{B}_1|$, where $\mathcal{B}_1$ is the set of all $(k-1)$-element subsets of $[n-1]$.
- $\binom{n-1}{k} = |\mathcal{B}_2|$, where $\mathcal{B}_2$ is the set of all k-element subsets of $[n-1]$.
- $\binom{n-1}{k-1} + \binom{n-1}{k} = |\mathcal{B}_1| + |\mathcal{B}_2| = |\mathcal{B}_1 \cup \mathcal{B}_2|$, since B_1 and B_2 have no common elements.

Now we try to give a bijection $\phi : \mathcal{A} \to \mathcal{B}_1 \cup \mathcal{B}_2$: this will show that $|\mathcal{A}| = |\mathcal{B}_1| + |\mathcal{B}_2|$, which is precisely the recurrence formula $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.

We define the bijection ϕ by specifying the output of an element of $\mathcal{A}$, that is of k-element subsets $S \subset [n]$. Let $\phi(S) = S' = S \setminus \{n\}$, meaning we remove n from S if it is present, and leave $S' = S$ otherwise. We thus produce $S' \in \mathcal{B}_1 \cup \mathcal{B}_2$, a subset of $[n-1]$ with either $k-1$ or k elements.

For example, the bijection ϕ for $\binom{4}{2} = \binom{3}{1} + \binom{3}{2}$ is given in the table, where $S' = \phi(S)$:

$S \in \mathcal{A}$	$\{1,2\}$	$\{1,3\}$	$\{1,4\}$	$\{2,3\}$	$\{2,4\}$	$\{3,4\}$
$S' \in \mathcal{B}_1$			$\{1\}$		$\{2\}$	$\{3\}$
$\in \mathcal{B}_2$	$\{1,2\}$	$\{1,3\}$		$\{2,3\}$		

Problem 4. Illustrate the mapping ϕ in the case of $\binom{5}{3} = \binom{4}{2} + \binom{4}{3}$. Make a table like the one above.

Problem 5. Formally define the inverse mapping $\psi : \mathcal{B}_1 \cup \mathcal{B}_2 \to \mathcal{A}$, which undoes ϕ. That is, given a subset $S' \subset [n-1]$ with either $k-1$ or k elements, define the corresponding k-element $S \subset [n]$. Define $S = \psi(S')$, with separate cases for $S' \in \mathcal{B}_1$ and $S' \in \mathcal{B}_2$.

You do not need to prove that your ψ is inverse to ϕ, but take a fairly large example of S, and verify that $\psi(\phi(S)) = S$; also take an example of S' and verify that and $\phi(\psi(S')) = S'$.