Exam topics

1. Basic structures: sets, lists, functions
   (a) Sets { }: write all elements, or define by condition
   (b) Set operations: $A \cup B$, $A \cap B$, $A \setminus B$, A^c
   (c) Lists ( ): Cartesian product $A \times B$
   (d) Functions $f : A \to B$ defined by any input-output rule
   (e) Injective function: $\forall a_1, a_2 \in A$: $a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2)$
   (f) Surjective function: $\forall b \in B$, $\exists a \in A$ with $f(a) = b$
   (g) A, B have same cardinality: there is a bijection $f : A \to B$
   (h) A is countable: there is a bijection $f : \mathbb{N} \to A$
2. Formal logic
   (a) Statements: definitely true or false
   (b) Conditional (open) statement $P(x)$: true/false depends on variable x
   (c) Logical operations: *and*, *or*, *not*, *implies*
   (d) Truth tables and logical equivalence
   (e) Implication $P \Rightarrow Q$ equivalent to: contrapositive $not(Q) \Rightarrow not(P)$;
   independent from: converse $Q \Rightarrow P$; inverse $not(P) \Rightarrow not(Q)$
   (f) Negate implication: $not(P \Rightarrow Q)$ is equivalent to: P *and* $not(Q)$
   (g) Quantifiers: $\forall$ for all , $\exists$ there exists;
   (h) Negate quantifiers: $not(\forall x, P(x))$ is equivalent to: $\exists x, not(P(x))$
   (i) Logical equivalences and set equations
   (j) Logic in mathematical language versus everyday language
3. Methods of proof (can be combined)
   (a) Direct proof
   (b) Proof by cases
   (c) Proof of the contrapositive
   (d) Proof by contradiction
   (e) Proof by induction (also complete induction)
4. Axioms of a Group $(G, *)$ (All variables below mean elements of G.)
   (a) Closure: $a * b \in G$.
   (b) Associativity: $(a * b) * c = a * (b * c)$
   (c) Identity: There is e with $e * a = a$ and $a * e = a$ for all a.
   (d) Inverses: For each a, there is some b with $a * b = e$ and $b * a = e$.
   Extra axioms
   (e) Commutativity: $a * b = b * a$.
   (f) Distributivity of times over plus: $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(b + c) \cdot a = b \cdot a + c \cdot a$.
5. Divisibility of integers (All variables below mean integers.)
   (a) Divisibility: $a | b$ means $b = ac$ for some $c \in \mathbb{Z}$

(b) Properties of divisibility:
- $a \mid b, c \implies a \mid mb{+}nc$ for all m, n
- $a|b$ and $b|c \implies a|c$.
- $a|b$ and $b|a \implies a = \pm b$.

(c) Prime and composite
- Test: a is composite $\implies a$ has prime factor $p \leq \sqrt{a}$.

(d) Greatest common divisor $\gcd(a, b)$; relatively prime means $\gcd(a, b) = 1$.

(e) Division Lemma: $a = qb + r$ with remainder $0 \leq r < b$.

(f) Euclidean Algorithm computes remainders $a > b > r_1 > \cdots > r_k > 0$.
- Computes $\gcd(a, b) = r_k$.
- Finds m, n with $\gcd(a, b) = ma + nb$.

(g) Consequences of $\gcd(a, b) = ma + nb$
- Find integer solutions (x, y) to equation $ax + by = c$, if $\gcd(a, b) \mid c$.
- If $e|a$ and $e|b$, then $e| \gcd(a, b)$.
- Euclid's Lemma: If $c \mid ab$ and $\gcd(c, a) = 1$, then $c|b$.
- Prime Lemma: If p is prime with $p \mid ab$, then $p|a$ or $p|b$.
- For $\bar{a} \in \mathbb{Z}_n$, find multiplicative inverse $\bar{b} = \bar{a}^{-1}$, i.e $ab \equiv 1 \pmod{n}$.

(h) Fundamental Theorem of Arithmetic
- $n > 1$ is a product of primes uniquely, except for rearranging factors.
- There is a unique list of powers $s_1, s_2, s_3, \ldots \geq 0$ with: $n = 2^{s_1} 3^{s_2} 5^{s_3} 7^{s_4} 11^{s_5} \cdots$.

6. Equivalence relation $\cong$ on a set S

(a) Defining properties:
- Reflexive: $a \equiv a$
- Symmetric: If $a \equiv b$, then $b \equiv a$.
- Transitive: If $a \equiv b$ and $b \equiv c$, then $a \equiv c$

(b) Equivalence class $[a] = \{b \in S \mid b \cong a\}$. Following are logically the same:
- $a \cong b$
- $a \in [b]$
- $[a] = [b]$, the same set

7. Clock arithmetic $\mathbb{Z}_n$

(a) Modular equivalence: $a \equiv b \pmod{n}$ means $n \mid a{-}b$. Class $\overline{a} = [a]$.

(b) Equivalence class $\overline{a} = [a]$. $\mathbb{Z}_n = \{\overline{0}, \overline{1}, \overline{2}, \ldots, \overline{n{-}1}\}$

(c) Modular addition and multiplication satisfy all usual rules of algebra

(d) Modular division: $\overline{a}^{\,-1} = \overline{b}$, where $\overline{a}\overline{b} = \overline{1}$, provided $\gcd(a, n) = 1$.

(e) In $\mathbb{Z}_p$ with p prime, every $\overline{a} \neq \overline{0}$ has $\overline{a}^{\,-1} \in \mathbb{Z}_p$.

8. Limits

(a) Real number axioms: commutative group axioms for $+, \cdot$; distribuitive law' axioms of order $<$.

(b) Completeness: If $S \subset \mathbb{R}$ has upper bound, then $\mathrm{lub}(S) = \sup(S) \in \mathbb{R}$.

(c) Convergent sequence $\lim\limits_{n\to\infty} a_n = \ell$: $\forall \epsilon{>}0, \exists N,\ n \geq N \Rightarrow |a_n{-}\ell| < \epsilon$

(d) Divergent sequence (a_n): $\forall \ell, \exists\, \epsilon{>}0, \forall N, \exists\, n{\geq}N$ with $|a_n{-}\ell| \geq \epsilon$.

(e) Infinite limit $\lim\limits_{n\to\infty} a_n = \infty$: $\forall B, \exists N,\ n \geq N \Rightarrow a_n > B$.

(f) Thm: If (a_n) increasing bounded sequence, then (a_n) convergent.

1. Decide whether the following statements are true or false. Prove the true ones and provide counterexamples for the false ones.

   (a) If a_n converges, then a_n/n converges.

   (b) If a_n converges and b_n is bounded, then $a_n b_n$ converges.

   (c) If $a_n \to \infty$ and $b_n \to -\infty$ as $n \to \infty$, then $a_n + b_n \to 0$ as $n \to \infty$.

   (d) If $a_n \to 0$ and $b_n \to 1$ as $n \to \infty$, then $b_n/a_n \to \infty$ as $n \to \infty$.

2. Suppose that $\{a_n\}$ is bounded. Prove that $a_n/n^k \to 0$, as $n \to \infty$ for all $k \in \mathbb{N}$.

3. Using the formal definition of the limit proof that if $\lim\limits_{n\to\infty} a_n = 1$ then $\lim\limits_{n\to\infty} \dfrac{a_n^2 - e}{a_n} = 1 - e$.

4. (AC) Let S be the set of all functions $f : \mathbb{N} \to \mathbb{N}$. Define a relation on S by letting $f \sim g$ if and only if $f(n) = g(n)$ for infinitely many n. Is this an equivalence relation? If so describe the equivalence classes.

5. (AC) Prove (assuming basic results of calculus) that $\int_0^\infty x^n e^{-x} dx = n!$.

6. (AC) For a function $f : \mathbb{R} \to \mathbb{R}$, define $\lim_{x\to c} f(x) = L$ to mean that $\forall \epsilon > 0 \; \exists \, \delta > 0$ such that $\forall \, x \in \mathbb{R}, |x - c| < \delta \Rightarrow |f(x) - L| < \epsilon$. Define the function $f : \mathbb{R} \to \mathbb{R}$ by

$$s(x) = \begin{cases} 0 & : x \le 0 \\ 1 & : x > 0 \end{cases}$$

   Prove by negating the definition of limit that it is not true that $\lim_{x\to 0} s(x) = 0$.

7. (a) Use a multiplication table to find all values $a \in \mathbb{Z}_7$ for which the equation

$$x^2 = a$$

   has a solution $x \in \mathbb{Z}_7$. For each such a, list all of the solutions x.

   (b) Find all solutions $x \in \mathbf{Z}_7$ to the equation $x^2 + \overline{2}x + \overline{6} = \overline{0}$.

8. Use quantifiers to express what it means for a sequence $(x_n)_{n\in\mathbb{N}}$ to *diverge*. You cannot use the terms *not* or *converge*.

9. Suppose $A, B \subseteq \mathbb{R}$ are bounded and non-empty. Show that $\sup(A \cup B) = \max\{\sup(A), \sup(B)\}$.

10. Suppose $S \subseteq \mathbb{R}$ is bounded and non-empty. Define a new set $3S$ by $3S = \{3x \mid x \in S\}$. Show that $\sup(3S) = 3\sup(S)$.

11. Let A, B be sets, and suppose there is a surjection $f : A \to B$. Prove that there is an injection $g : B \to A$.

12. (a) Define $x \in \mathbb{R}$ to be a **linear algebraic number** if there are integers $a, b \in \mathbb{Z}$, with $a \neq 0$, such that $ax + b = 0$. Prove that the set of linear algebraic numbers is countable. *Hint: Construct an injection into the set $\mathbb{Z}^2$.*

    (b) Define $x \in \mathbb{R}$ to be a **quadratic algebraic number** if there are integers $a, b, c \in \mathbb{Z}$, with $a \neq 0$, such that $ax^2 + bx + c = 0$. Prove that the set of quadratic algebraic numbers is countable. *Hint: Construct an injection into the set $\mathbb{Z}^3$.*

13. Use the formal definition of limit to prove the following.

    (a) $\displaystyle\lim_{n\to\infty} \frac{n^2+3}{2n^3-4} = 0$

    (b) $\displaystyle\lim_{n\to\infty} \frac{4n-5}{2n+7} = 2$

    (c) $\displaystyle\lim_{n\to\infty} \frac{n^3-3n}{n+5} = +\infty$

    (d) $\displaystyle\lim_{n\to\infty} \frac{n^2-7}{1-n} = -\infty$

14. For each of the following, determine if $\sim$ defines an equivalence relation on the set S. If it does, prove it and describe the equivalence classes. If it does not, explain why.

    (a) $S = \mathbb{R} \times \mathbb{R}$. For (a, b) and $(c, d) \in S$, define $(a, b) \sim (c, d)$ if $3a + 5b = 3c + 5d$.

    (b) $S = \mathbb{R}$. For $a, b \in S$, $a \sim b$ if $a < b$.

    (c) $S = \mathbb{Z}$. For $a, b \in S$, $a \sim b$ if $a \mid b$.

    (d) $S = \mathbb{R} \times \mathbb{R}$. For (a, b) and $(c, d) \in S$, define $(a, b) \sim (c, d)$ if $\lceil a \rceil = \lceil c \rceil$ and $\lceil b \rceil = \lceil d \rceil$. Here $\lceil x \rceil$ is the smallest integer greater than or equal to x.

15. Consider Z_n.

    (a) Under what conditions on n does every nonzero element have a multiplicative inverse? How about an additive inverse?

    (b) Does every nonzero element have a multiplicative inverse in Z_{21}?

    (c) Does 5 have a multiplicative inverse in Z_{21}? Explain why or why not. If it does, find 5^{-1}.

    (d) Solve the equation $5x - 14 = 19$ in $\mathbb{Z}_{21}$.

16. Let $A = \{a, b, c\}$ and $B = \{x, y\}$. List all elements of

    (a) $A \cup B$

    (b) $A \cap B$

    (c) $A \backslash B$

(d) $A \times B$

(e) Power set of A

17. Let $S(n) = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid max\{x, y\} = n\}$. Prove that $S(3) \cap S(5)$ is the empty set.

18. Let A and B be sets with n elements. Show that any injective function from A to B is surjective as well using induction on n.

19. Let $f : \mathbb{N} \to \mathbb{N}$, given by $f(n) = |n - 4|$.

    (a) Prove that f is surjective

    (b) Prove that f is not injective

20. Let $f : A \to B$ and $g : B \to A$ be functions satisfying $f(g(x)) = x$ for all $x \in B$. Prove that f is surjective.

21. Let X be a set with n elements and $B = \{p, q\}$. Find the number of surjective functions from X to B.

22. Describe a concrete bijection from $\mathbb{N}$ to $\mathbb{N} \times \{1, 2, 3\}$. Briefly tell why it is injective and surjective.

1. Decide whether the following statements are true or false. Prove the true ones and provide counterexamples for the false ones.

   (a) If a_n converges, then a_n/n converges.

Solution. This seems reasonable so we will try to prove that it is true. In fact it would make sense that if a_n converges then a_n/n should converge to 0. So this is what we will try to show.

Goal: to find good N_ϵ so that:

$$n \geq N_\epsilon \implies \left|\frac{a_n}{n} - 0\right| < \epsilon$$

So now please consider:

$$\begin{aligned}
\left|\frac{a_n}{n} - 0\right| &= \left|\frac{a_n - a}{n} + \frac{a}{n}\right| && \text{(Clever splitting of 0)} \\
&\leq \left|\frac{a_n - a}{n}\right| + \left|\frac{a}{n}\right| && \text{(Triangle Ineq.)} \\
&= |a_n - a|\frac{1}{n} + |a|\frac{1}{n} && (n \text{ is } +)
\end{aligned}$$

Here is where we need to make our good choice for N_ϵ Let's choose $N_\epsilon = \max\{2, N'_\epsilon, N''_\epsilon\}$ where:

By the definition of a_n converging we can choose N'_ϵ so that :

$$n \geq N'_\epsilon \implies \boxed{|a_n - a| < \epsilon}$$

By since $\dfrac{1}{n} \to 0$ we can choose N''_ϵ so that :

$$n \geq N''_\epsilon \implies \left|\frac{1}{n} - 0\right| = \boxed{\frac{1}{n} < \frac{\epsilon}{2|a|}}$$

And therefore we get if $n \geq N_\epsilon$ then:

$$\begin{aligned}
\left|\frac{a_n}{n} - 0\right| &\leq |a_n - a|\frac{1}{n} + |a|\frac{1}{n} && \text{(From before)} \\
&< (\epsilon)\frac{1}{(2)} + |a|\left(\frac{\epsilon}{2|a|}\right) && \text{(Substitutions)} \\
&= \frac{\epsilon}{2} + \frac{\epsilon}{2} && \text{(Simplify)}
\end{aligned}$$

And so,

$$\left|\frac{a_n}{n} - 0\right| < \epsilon$$

So therefore by the definition of convergence $\dfrac{a_n}{n}$ **converges** (to 0). And so this statement is $\boxed{\text{TRUE}}$.

Lastly I should note that the choosing of N_ϵ takes some time and lots of scrap paper to work out. There is an easier way! If you use the theorem that says $\boxed{\text{Every convergent sequence is bounded}}$ it is very straightforward. Try it out (Or look at the proof of 2)!

(b) If a_n converges and b_n is bounded, then $a_n b_n$ converges.

Solution. Let's review with the definition of bounded:

Definition:
$$\{x_n\} \text{ is } \textbf{bounded} \quad \Leftrightarrow \quad \exists\, M > 0 \text{ such that } x_n \leq M \;\forall\, n \in \mathbb{N}$$

Let's try for a counterexample. Let's take some vary simple choices:

$a_n = 1 = \{1, 1, 1, \ldots\}$

$b_n = (-1)^n = \{-1, 1, -1, \ldots\}$

Then a_n obviously converges (to 1) and b_n is bounded by $M = 1$. and yet:

$a_n b_n = 1(-1)^n = (-1)^n$ which does not converge.

And so this statement is $\boxed{\text{FALSE}}$.

(c) If $a_n \to \infty$ and $b_n \to -\infty$ as $n \to \infty$, then $a_n + b_n \to 0$ as $n \to \infty$.

Solution. This should not be true. because things can head to $\pm\infty$ at different rates. Let's try for a counter-example. Let's choose:

$a_n = 2n$

$b_n = -n$

Then we can see that these meet are criteria of $a_n \to \infty$ and $b_n \to -\infty$ and yet:

$$a_n + b_n = 2n + (-n) = \boxed{n \to \infty \neq 0}$$

And so this statement is $\boxed{\text{FALSE}}$.

(d) If $a_n \to 0$ and $b_n \to 1$ as $n \to \infty$, then $b_n/a_n \to \infty$ as $n \to \infty$.

Solution. The claim is that this is FALSE and this will be because we could have $a_n \to 0$ and yet $a_n < 0\ \forall n$ which would yield $b_n/a_n \to -\infty$.

So let's take

$b_n = 1 \to 1$

$a_n = -\dfrac{1}{n} \to 0$

Then we can look at:

$b_n/a_n = 1/\left(-\dfrac{1}{n}\right) = \boxed{-n \to -\infty \neq \infty}$

And so this statement is $\boxed{\text{FALSE}}$.

Let's quickly take some practice formally showing that $-n \to -\infty$. Let's first recall a definition of what we mean by a sequence $x_n \to \pm\infty$.

> **Definition:** $x_n \to \infty$ means:
> $$\forall M \in \mathbb{R},\ \exists N_M \in \mathbb{N} \text{ such that: } n \geq N_M \implies x_n > M.$$
>
> **Definition:** $x_n \to -\infty$ means:
> $$\forall M \in \mathbb{R},\ \exists N_M \in \mathbb{N} \text{ such that: } n \geq N_M \implies x_n < M.$$

So it is now clear what we have to do:

> **Goal:**to choose a good N_M so that:
> $$n \geq N_M \implies -n < M$$

Since $M \in \mathbb{R}$ we need to do a little extra work to make sure that we choose $N_M \in \mathbb{N}$ but still it is not so bad.

Let's try: $N_M = \lceil |M| \rceil + 1$ (the ceiling function where we just round up of the absolute value.) So now

$n \geq \lceil |M| \rceil + 1 \implies -n \leq \lceil |M| \rceil - 1 < M$

And so we have $\boxed{n \geq N_M \implies -n < M}$ so by the definition: $\boxed{-n \to -\infty}$

> Honestly though I had to sit and play with this for quite some time to see that $N_M = \lceil |M| \rceil + 1$ was a good choice. To help me I considered what if $M = -4.5$ or what if $M = 4.5$ to help me:
>
> If $M = -4.5$ then we get
> $N_M = \lceil |-4.5| \rceil + 1 = \lceil 4.5 \rceil + 1 = 5 + 1 = 6$ so we get $n \geq 6 \implies -n < -4.5$
>
> Or if $M = 4.5$ then we get
> $N_M = \lceil |4.5| \rceil + 1 = \lceil 4.5 \rceil + 1 = 5 + 1 = 6$ so we get $n \geq 6 \implies -n < 4.5$
>
> Choosing nice specific examples like this can help craft good choices of N_M (or δ or N_ϵ) on occasion.

2. Suppose that $\{a_n\}$ is bounded. Prove that $a_n/n^k \to 0$, as $n \to \infty$ for all $k \in \mathbb{N}$.

Solution. The first step is to preform some simplifications. Let's apply the following theorem

> **Theorem:**
>
> $$x_n \to 0 \quad \Leftrightarrow \quad |x_n| \to 0$$

Which is a result of the Squeeze Theorem. So therefore $a_n/n^k \to 0 \Leftrightarrow |a_n|/n^k \to 0$.

Now let's apply the Squeeze Theorem directly which states:

> **Theorem:** (Squeeze)
>
> If $x_n \to a$ and $y_n \to a$ and if $x_n \le w_n \le y_n \quad \forall\, n \ge N_0$ then:
>
> $$w_n \to a$$

Our desired thing to show converges is $|a_n|/n^k \to 0$. So let's make good choice for $x_n \to 0$ and $y_n \to 0$ so that the Squeeze Theorem will help us.

Since we have $0 \le |a_n|/n^k$ we can take $\boxed{x_n = 0 \to 0}$. Now for y_n let's use the fact that a_n bounded, which means that:

$$\exists\, M > 0 \text{ such that } a_n \le M \;\forall\, n \in \mathbb{N}$$

Also if $n, k \ge 1$ then $n \le n^k \implies \dfrac{1}{n^k} \le \dfrac{1}{n}$. Comboing these we have that $|a_n|/n^k \le M/n^k \le M/n$.
So therefore we should choose $\boxed{y_n = M/n}$ which converges to 0 by the multiplicative rule of limits

$$\frac{M}{n} = M\frac{1}{n} \to M(0) = 0$$

By the Squeeze theorem we win!

3. Using the formal definition of the limit proof that if $\lim_{n\to\infty} a_n = 1$ then $\lim_{n\to\infty} \frac{a_n^2 - e}{a_n} = 1 - e$.

Solution. This would be much easier if we could use properties of limits. Let's try it out using the definition though:

Goal: to find good N_ϵ so that:

$$n \geq N_\epsilon \implies \left|\frac{a_n^2 - e}{a_n} - (1-e)\right| < \epsilon$$

We have in our tool belt that $a_n \to 1$ which means:

$$\forall \epsilon' > 0\ \exists N'_\epsilon \in \mathbb{N} \text{ so that: } n \geq N'_\epsilon \implies |a_n - 1| < \epsilon' \tag{**}$$

So let's rearrange $\left|\frac{a_n^2 - e}{a_n} - (1-e)\right|$ a bit and see if there is any natural progressions!

$$\begin{aligned}
\left|\frac{a_n^2 - e}{a_n} - (1-e)\right| &= \left|\frac{a_n^2 - e - a_n + ea_n}{a_n}\right| && \text{(Common Denominator)}\\
&= \left|\frac{a_n^2 + (e-1)a_n - e}{a_n}\right| && \text{(Rearrange to look like Quadratic)}\\
&= \left|\frac{(a_n + e)(a_n - 1)}{a_n}\right| && \text{(Factor)}\\
&= \left|\frac{a_n + e}{a_n}\right| |a_n - 1| && \text{(Property of } |\cdot|)
\end{aligned}$$

Here we need to employ some trick to bound $\left|\frac{a_n + e}{a_n}\right|$. Let's consider N''_ϵ so that $n \geq N''_\epsilon \implies |a_n - 1| < .5 \implies .5 < a_n < 1.5$.

Now to make it look like $\left|\frac{a_n + e}{a_n}\right|$.

$.5 < a_n < 1.5 \implies 0 < .5 + e < a_n + e < 1.5 + e \implies 0 < \frac{a_n + e}{a_n} < \frac{1.5 + e}{.5} = 3 + 2e \implies \boxed{\left|\frac{a_n + e}{a_n}\right| < 3 + 2e}$

So let's consider choosing N'_ϵ from (**) so that:

$$n \geq N'_\epsilon \implies \boxed{|a_n - 1| < \frac{\epsilon}{3 + 2e}}$$

Finally taking $N_\epsilon = \max\{N'_\epsilon, N''_\epsilon\}$ gives us:

$$n \geq N_\epsilon \implies \left|\frac{a_n + e}{a_n}\right| |a_n - 1| < (3 + 2e)\left(\frac{\epsilon}{3 + 2e}\right) = \epsilon$$

Making us the winners!

4. (AC) Let S be the set of all functions $f : \mathbb{N} \to \mathbb{N}$. Define a relation on S by letting $f \sim g$ if and only if $f(n) = g(n)$ for infinitely many n. Is this an equivalence relation? If so describe the equivalence classes.

   Solution. The relation is reflexive since for any $f : \mathbb{N} \to \mathbb{N}$, $f(n) = f(n)\ \forall\, n \in \mathbb{N}$ where $\mathbb{N}$ is an infinite set. The relation is symmetric since $=$ is symmetric. In fact, the relation is not transitive. To see why, consider the following functions. Let $f(n)$ be equal to 1 for n even and 2 for n odd. Let $g(n)$ be equal to 3 for n even and 2 for n odd. Let $h(n)$ be equal to 3 for n even and 4 for n odd. Then $f \sim g$ and $g \sim h$ but f and h agree nowhere.

5. (AC) Prove (assuming basic results of calculus) that $\int_0^\infty x^n e^{-x} dx = n!$.

   Solution. We proceed by induction on $n \geq 0$. The base case holds since

$$\begin{aligned} \int_0^\infty e^{-x} dx &= \lim_{b\to\infty} \int_0^b e^{-x} dx \\ &= \lim_{b\to\infty} -e^{-x}|_0^b \\ &= \lim_{b\to\infty} -e^{-b} + 1 \\ &= 1 \\ &= 0! \end{aligned}$$

To show that the inductive step holds, we integrate by parts with $u = x^n, dv = e^{-x}dx$. We have

$$\begin{aligned} \int_0^\infty x^n e^{-x} dx &= \lim_{b\to\infty} \int_0^b x^n e^{-x} dx \\ &= \lim_{b\to\infty} \left[[x^n(-e^{-x})]_0^b - \int_0^b n x^{n-1}[-e^{-x}]dx \right] \\ &= 0 + n \int_0^\infty x^{n-1} e^{-x} dx \\ &= n(n-1)! \qquad\qquad \text{by the inductive hypothesis} \\ &= n!. \end{aligned}$$

Thus by induction, the identity holds for all $n \geq 0$.

6. (AC) For a function $f : \mathbb{R} \to \mathbb{R}$, define $\lim_{x\to c} f(x) = L$ to mean that $\forall \epsilon > 0\ \exists\, \delta > 0$ such that $\forall\, x \in \mathbb{R}, |x - c| < \delta \Rightarrow |f(x) - L| < \epsilon$. Define the function $f : \mathbb{R} \to \mathbb{R}$ by

$$s(x) = \begin{cases} 0 & : x \le 0 \\ 1 & : x > 0 \end{cases}$$

Prove by negating the definition of limit that it is not true that $\lim_{x\to 0} s(x) = 0$.

Solution. The negation of $\lim\limits_{x\to 0} s(x) = 0$ is

$$\exists\, \epsilon > 0 \text{ such that } \forall\, \delta > 0, \exists x \in \mathbb{R} \text{ such that } |x - 0| < \delta \text{ and } |s(x) - 0| \ge \epsilon$$

Letting $\epsilon = 1/2$, then $\forall\, \delta > 0, \delta/2 \in \mathbb{R}$ with $|\delta/2 - 0| = \delta/2 < \delta$ and $|s(\delta/2) - 0| = |1 - 0| = 1 \ge 1/2$. Thus the limit is not equal to 0.

7. (a) Use a multiplication table to find all values $a \in \mathbb{Z}_7$ for which the equation

$$x^2 = a$$

has a solution $x \in \mathbb{Z}_7$. For each such a, list all of the solutions x.

Solution. We only need to look at the diagonal of the multiplication table for $\mathbb{Z}_7$. Then the equation $x^2 = a$ has a solution $x \in \mathbb{Z}_7$ if and only if $a \in \left\{\overline{0}, \overline{1}, \overline{2}, \overline{4}\right\}$. When $a = \overline{0}$, the only solution is $x = \overline{0}$. When $a = \overline{1}$, the solutions are $x = \overline{1}$ and $x = \overline{6}$. When $a = \overline{2}$, the solutions are $x = \overline{3}$ and $x = \overline{4}$. When $a = \overline{4}$, the solutions are $x = \overline{2}$ and $x = \overline{5}$.

(b) Find all solutions $x \in \mathbf{Z}_7$ to the equation $x^2 + \overline{2}x + \overline{6} = \overline{0}$.

Solution. Adding $\overline{2}$ to both sides, the given equation is equivalent to $x^2 + \overline{2}x + \overline{1} = \overline{2}$. We can factor the left-hand side to get

$$(x + \overline{1})^2 = \overline{2}.$$

It follows from part (a) that $x + \overline{1} = \overline{3}$ or $x + \overline{1} = \overline{4}$, and hence $x = \overline{2}$ or $x = \overline{3}$.

8. Use quantifiers to express what it means for a sequence $(x_n)_{n\in\mathbb{N}}$ to *diverge*. You cannot use the terms *not* or *converge*.

Solution. A sequence $(x_n)_{n\in\mathbb{N}}$ *diverges* if for every $L \in \mathbb{R}$ there is some $\epsilon > 0$ such that for all $N \in \mathbb{N}$ there is some natural number $n \geq N$ for which $|x_n - L| \geq \epsilon$. In terms of quantifiers this is

$$\forall L \in \mathbb{R}\ \exists \epsilon > 0\ \forall N \in \mathbb{N}\ \exists n \geq N, |x_n - L| \geq \epsilon.$$

9. Suppose $A, B \subseteq \mathbb{R}$ are bounded and non-empty. Show that $\sup(A \cup B) = \max\{\sup(A), \sup(B)\}$.

Solution. First note that since A and B are both bounded and non-empty, the same is true of $A \cup B$ and so $\sup(A \cup B) \in \mathbb{R}$ exists. It follows immediately from Beck Proposition 8.50 that $\sup(A \cup B) \geq \max\{\sup(A), \sup(B)\}$, since A and B are both subsets of $A \cup B$. We need to prove the reverse inequality. For sake of contradiction, suppose $\sup(A \cup B) > \max\{\sup(A), \sup(B)\}$. Then neither $\sup(A)$, nor $\sup(B)$ can be an upper bound for $A \cup B$. So there is some $x \in A \cup B$ with $x > \sup(A)$ and $x > \sup(B)$. But $x \in A \cup B$ implies that $x \in A$ or $x \in B$, so this cannot happen.

10. Suppose $S \subseteq \mathbb{R}$ is bounded and non-empty. Define a new set $3S$ by $3S = \{3x \mid x \in S\}$. Show that $\sup(3S) = 3\sup(S)$.

Solution. First we will show $\sup(3S) \leq 3\sup(S)$. Let $y \in 3S$. Then $y = 3x$ for some $x \in S$. Since $\sup(S)$ is an upper bound for S, we have $x \leq \sup(S)$. This gives $y = 3x \leq 3\sup(S)$. Since this is true for all $y \in 3S$, it follows that $3\sup(S)$ is an upper bound for the set $3S$. Since $\sup(3S)$ is the *least* upper bound, we must have $\sup(3S) \leq 3\sup(S)$.

To prove the reverse inequality, let $x \in S$. Then $3x \in 3S$ and so $3x \leq \sup(3x)$. Equivalently, $x \leq 3^{-1}\sup(3x)$. It follows that $3^{-1}\sup(3x)$ is an upper bound for S and so $\sup(S) \leq 3^{-1}\sup(3x)$, since $\sup(S)$ is the *least* upper bound. Multiplying both sides by 3 proves the result.

11. Let A, B be finite sets, and suppose there is a surjection $f : A \to B$. Prove that there is an injection $g : B \to A$ such that $f \circ g : B \to B$ is the identity function.

Solution. Let $b \in B$, we want to define $g(b) \in A$. Since f is surjective, it follows that $f^{-1}(b)$ is a non-empty set. Let $a \in f^{-1}(b)$ be any element of this set, and declare $g(b) = a$. We obviously have that $f \circ g$ is the identity, since $f(g(b)) = f(a) = b$ for all $b \in B$. To show g is injective, suppose there are $b, b' \in B$ with $g(b) = g(g')$. Let $a = g(b)$ denote this common value. Then by construction of g, we have $a \in f^{-1}(b)$ and $a \in f^{-1}(b')$. Applying f to a therefore gives $f(a) = b$ and $f(a) = b'$, and so $b = b'$.

12. (a) Define $x \in \mathbb{R}$ to be a **linear algebraic number** if there are integers $a, b \in \mathbb{Z}$, with $a \neq 0$, such that $ax + b = 0$. Prove that the set of linear algebraic numbers is countable. *Hint: Construct an injection into the set* $\mathbb{Z}^2$.

    Solution. Since $\mathbb{Z}^2$ is countable, it suffices to show that there is an injection from the set of linear algebraic numbers into $\mathbb{Z}^2$. Given a linear algebraic number x, there are $a, b \in \mathbb{Z}$ with $ax + b = 0$, and $a \neq 0$. (Note that there will be multiple choices of a, b for which $ax + b = 0$. However, we can choose a, b canonically be requiring $a > 0$ and $\gcd(a, b) = 1$.) Then we define the function by sending x to these values of a, b. To see that this function is injective, suppose there are $a, b \in \mathbb{Z}$ with $a \neq 0$ and $ax + b = ay + b$ for some $x, y \in \mathbb{R}$. Then obviously $x = y$, since $a \neq 0$. (Note that the linear algebraic numbers are exactly the rational numbers, and we are just proving that $\mathbb{Q}$ is countable.)

    (b) Define $x \in \mathbb{R}$ to be a **quadratic algebraic number** if there are integers $a, b, c \in \mathbb{Z}$, with $a \neq 0$, such that $ax^2 + bx + c = 0$. Prove that the set of quadratic algebraic numbers is countable. *Hint: Construct an injection into the set* $\mathbb{Z}^3$.

    Solution. We will show that there is an injective function from the set of quadratic algebraic numbers into $\mathbb{Z}^3$. Given a quadratic algebraic number x, there are $a, b, c \in \mathbb{Z}$ with $ax^2 + bx + c = 0$, and $a \neq 0$. As in part (a), the basic idea is to send x to $(a, b, c) \in \mathbb{Z}^3$. However, there is an extra feature of this problem that makes it a little more difficult, and this is coming from the fact that most quadratic equations have 2 solutions (this will mean that our function will not be injective, unless we are careful). Here is a way to define the function so that it is injective. Let x be as above. If there is some $x' \in \mathbb{R}$ with $a(x')^2 + bx' + c = 0$ and $x \neq x'$, then without loss of generality we may suppose $x' < x$. Send x to (a, b, c) where $a > 0$ and a, b, c have no common divisors, and send x' to $(-a, -b, -c)$. (Note that by the quadratic formula there can be at most two solutions of $ax^2 + bx + c = 0$.)
    To see that this function is injective, suppose x_1 and x_2 are both sent to (a, b, c). This implies that x_1, x_2 are both solutions to the equation $ax^2 + bx + c = 0$. If $a > 0$, then by construction of our function, x_1 and x_2 are both equal to the solution with the maximum values and so $x_1 = x_2$. If $a < 0$, then x_1 and x_2 are both equal to the solution with the smallest value and so $x_1 = x_2$.

13. Use the formal definition of limit to prove the following.

    (a) $\displaystyle \lim_{n\to\infty} \frac{n^2+3}{2n^3-4} = 0$

Solution. Let $\varepsilon > 0$ be given, aribitrary. Define $N = \max\{3, \frac{2}{\varepsilon} + 2\}$. Let $n \geq N$, $n \in \mathbb{N}$ be arbitrary. Then,

$$\begin{aligned}
\left|\frac{n^2+3}{2n^3-4} - 0\right| &= \frac{n^2+3}{2n^3-4} && \text{(since } n \geq 3\text{)} \\
&\leq \frac{n^2+3n^2}{2n^3-4n^2} && \text{(We increased the numerator and decreased the denominator,} \\
& && \text{keeping in mind } 2n^3 - 2n^2 > 0, asn > 2\text{)} \\
&= \frac{2}{n-2} && \text{(Factor and cancel out common terms)} \\
&\leq \frac{2}{N-2} && (n \geq N) \\
&\leq \varepsilon && (N \geq \frac{2}{\varepsilon} + 2)
\end{aligned}$$

Thus, $\forall \varepsilon > 0$, $\exists N$ such that $\forall n > N$ with $n \in \mathbb{N}$, $\left|\frac{n^2+3}{2n^3-4} - 0\right| < \varepsilon$. Thus, indeed $\lim_{n\to\infty} \frac{n^2+3}{2n^3-4} = 0$.

(b) $\lim_{n\to\infty} \frac{4n-5}{2n+7} = 2$

Solution. Let $\varepsilon > 0$ be given, aribitrary. Define $N = \frac{1}{\varepsilon}$. Let $n \geq N$, $n \in \mathbb{N}$ be arbitrary. Then,

$$\begin{aligned}
\left|\frac{4n-5}{2n+7} - 2\right| &= \frac{19}{2n+7} && \text{(since } n > 0\text{)} \\
&< \frac{19}{2n} && \text{(We decreased the denominator,} \\
&< \frac{1}{n} && \\
&\leq \frac{1}{N} && (n \geq N) \\
&= \varepsilon && (N = \frac{1}{\varepsilon})
\end{aligned}$$

Thus, $\forall \varepsilon > 0$, $\exists N$ such that $\forall n > N$ with $n \in \mathbb{N}$, $\left|\frac{4n-5}{2n+7} - 2\right| < \varepsilon$. Thus, indeed $\lim_{n\to\infty} \frac{4n-5}{2n+7} = 0$.

(c) $\lim_{n\to\infty} \frac{n^3-3n}{n+5} = +\infty$

Solution. Let $M > 0$ be given, aribitrary. Define $N = \sqrt{6M+3}$. Let $n \geq N$, $n \in \mathbb{N}$ be arbitrary. Then,

$$\begin{aligned}
\frac{n^3-3n}{n+5} &\geq \frac{n^3-3n}{n+5n} && \text{(since } n \geq 1\text{)}\\
&= \frac{n^3-3n}{6n}\\
&= \frac{n^2-3}{6}\\
&\geq \frac{N^2-3}{6} && (n \geq N)\\
&= M && (N = \sqrt{6M+3})
\end{aligned}$$

Thus, $\forall M > 0$, $\exists N$ such that $\forall n > N$ with $n \in \mathbb{N}$, $\frac{n^3-3n}{n+5} \geq M$. Thus, indeed $\lim_{n\to\infty} \frac{n^3-3n}{n+5} = +\infty$.

(d) $\lim_{n\to\infty} \frac{n^2-7}{1-n} = -\infty$

Solution. Let $M < 0$ be given, aribitrary. Define $N = 7 - M$. Let $n \geq N$, $n \in \mathbb{N}$ be arbitrary. Then,

$$\begin{aligned}
\frac{n^2-7}{1-n} &< \frac{n^2-7}{-n} && \text{(since } n > 7,\ \text{thus } n^2-7 > 0\text{)}\\
&\leq \frac{n^2-7n}{-n} && \text{(since the denominator is negative and } n > 7,\\
& && \text{decreasing the numerator, while still keeping it positive)}\\
&= 7-n\\
&\leq 7-N && (n \geq N)\\
&= M && (N = 7-M)
\end{aligned}$$

Thus, $\forall M < 0$, $\exists N$ such that $\forall n > N$ with $n \in \mathbb{N}$, $\frac{n^2-7}{1-n} \leq M$. Thus, indeed $\lim_{n\to\infty} \frac{n^2-7}{1-n} = -\infty$.

14.

15.

16. Let $A = \{a, b, c\}$ and $B = \{a, x\}$. List all elements of

    (a) $A \cup B$

    (b) $A \cap B$

    (c) $A \backslash B$

    (d) $A \times B$

    (e) Power set of A

    Solution. $A \cup B = \{a, b, c, x\}$, $A \cap B = \{a\}$, $A \backslash B = \{b, c\}$, $A \times B = \{(a, a), (a, x), (b, a), (b, x), (c, a), (c, x)\}$, Power set of A is $\{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}\}$.

17. Let $S(n) = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid max\{x, y\} = n\}$. Prove that $S(3) \cap S(5)$ is the empty set.

18. Let A and B be sets with n elements. Show that any injective function from A to B is surjective as well using induction on n.

19. Let $f : \mathbb{N} \to \mathbb{N}$, given by $f(n) = |n - 4|$.

    (a) Prove that f is surjective

    (b) Prove that f is not injective

20. Let $f : A \to B$ and $g : B \to A$ be functions satisfying $f(g(x)) = x$ for all $x \in B$. Prove that f is surjective.

    Solution. First attempt: Assume f is not surjective. Then there is a $b \in B$ such that there are no $a \in A$ with $f(a) = b$. Let $c = g(b)$, then $f(c) = f(g(b)) = b$ by assumption, hence we found a $c \in A$ with $f(c) = b$ which contradicts with the assumption.
    Second attempt: Given $b \in B$, let $a = g(b)$, and compute $f(a) = f(g(b)) = b$ by assumption. Since b was arbitrary, this shows that f is surjective.

21. Let X be a set with n elements and $B = \{p, q\}$. Find the number of surjective functions from X to B.

    Solution. There are $2 \cdot 2 \cdot ... \cdot 2 = 2^n$ possible functions from X to B. Now list all functions that are not injective, you will find a small number ..., subtract it from 2^n to get the final result.

22. Describe a concrete bijection from $\mathbb{N}$ to $\mathbb{N} \times \{1, 2, 3\}$. Briefly tell why it is injective and surjective.

Solution. By division lemma, given n, there is a unique q and r with $0 \leq r < 3$ with $n = 3 \cdot q + r$, we could define $f(n)$ by $f(n) = (q, r+1)$. Then f has the inverse function given by $g(a, b) = 3 \cdot a + b$. Check that $f(g(a, b)) = (a, b)$ and $g(f(n)) = n$.