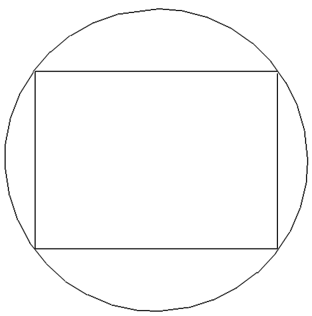
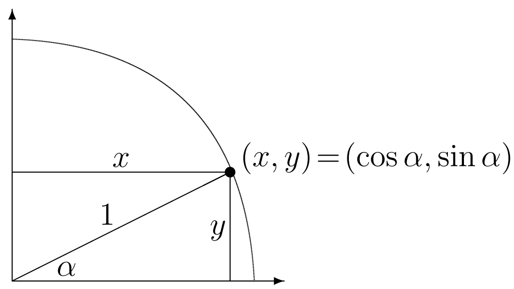
Math 254H **Weekly Homework 4** **Fall 2022**

Optimizing a Sawmill

We seek the most efficient ways to cut up round logs into rectangular planks and beams. We look at the cross-section of our log, which we assume to be a circle of radius 1 foot.

To begin, we want to find the dimensions of a single beam of maximum cross-section area cut out of this circle. It is convenient to look at a quadrant.

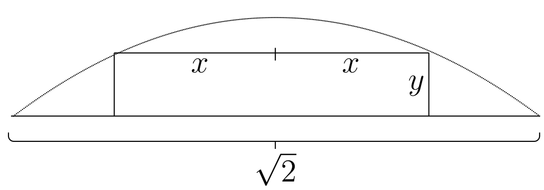




1. Consider the variable α, the angle between the x-axis and the corner point. Then the corner point has coordinates $(\cos\alpha, \sin\alpha)$, and we want to maximize the area function $f(\alpha) = \cos(\alpha)\sin(\alpha)$ on the interval $0 \leq \alpha \leq \pi/2$. (Here $f(\alpha)$ is $\frac{1}{4}$ of the beam's cross-section area.)

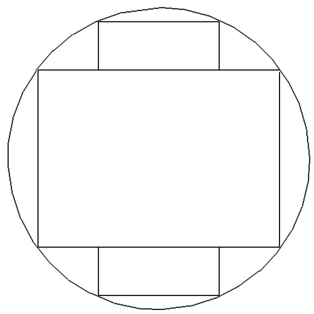
Solve this problem algebraically, showing that the optimal beam is *square*. In your computations, keep in mind the identities $\sin(2\theta) = 2\cos\theta\sin\theta$, $\cos(2\theta) = \cos^2\theta - \sin^2\theta$. What is special about the values of f at the boundary points $\alpha = 0, \pi/2$?

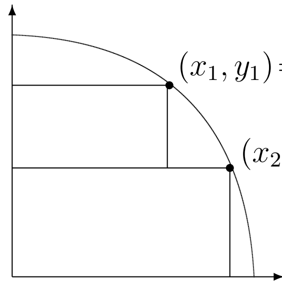
2. Suppose we have cut out the square beam described above, and we want to cut extra planks of maximum cross-section out of the remaining pieces.



That is, we maximize the area xy subject to the relation $x^2 + (y + \frac{\sqrt{2}}{2})^2 = 1$. Explain these equations, and find the maximum of the area function. (Do you need one- or two-variable calculus?)

3. Next consider simultaneously cutting out a thick beam and two planks above and below it.





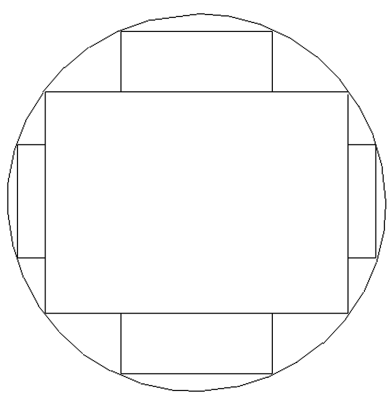
Write the area cut out of the quadrant as a function $f(\alpha, \beta)$ of the angles α, β marked on the circle, and maximize over the region $0 \le \alpha, \beta \le \pi/2$.

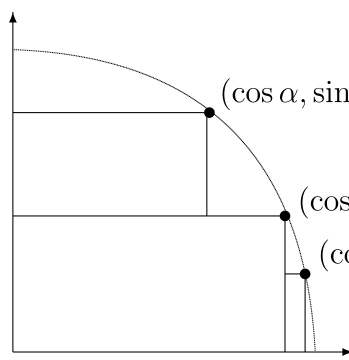
Include a printout of the contour plot of $f(\alpha, \beta)$ over this region, and a magnified view to approximate the maximum point(s) to two decimal places.

Also: is the result the same as successively cutting out a maximal beam and then maximal planks as in #1,2 above? Doesn't it *have* to be the same? Explain, referring to the contour plot.

Extra Credit: Solve the optimization equations algebraically (tricky trig).

4. Now the main problem. Consider a beam and four surrounding planks:





Write down the combined area of the wood cut out of the quadrant at right, as a function $f(\alpha, \beta, \gamma)$. Be careful to count each region only once, subtracting out overlaps as in Prob 3.

5. Maximize the three-variable area function from Prob 4. Here, as usually with practical problems, the only feasible method is the numerical Gradient Flow Method.

Start with a rough approximation (really a guess) for the maximum point, $\mathbf{v}_0 = (\alpha_0, \beta_0, \gamma_0)$, and compute $\nabla f(\mathbf{v}_0)$. If $\mathbf{v}_0$ were really the maximum, we would have $\nabla f(\mathbf{v}_0) = (0, 0, 0)$; but since it is only an estimate, the non-zero gradient points in the direction of maximum increase of $f(\mathbf{v})$. Obtain an improved estimate $\mathbf{v}_1 = \mathbf{v}_0 + \epsilon \nabla f(\mathbf{v}_0)$, where ϵ is a small fixed parameter, such as $\epsilon = 0.5$.

Iterate this process to get a sequence of approximations $\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \ldots$, converging to a true max point. If the algorithm takes too large steps, it might overshoot and jump back and forth around the maximum: if so, decrease ϵ. If it takes too small steps, it might converge very slowly: increase ϵ. A more serious problem is that it might find one max point near the initial estimate, and miss others. Try several initial values to see that they all converge to the same point.

Hand in a spreadsheet or computer algebra printout giving the maximum point correct to two decimal places.

Extra Credit: Compare the answer with Prob 2, and explain the coincidence.

6. Some theoretical questions about the Gradient Flow Method

a. How will the algorithm behave if there is no maximum point, and $f(\mathbf{v})$ keeps increasing in some direction?

b. How would you modify the algorithm to find a min point?

c. How would the algorithm behave close to a saddle point?

7. Re-do Problems 1 & 2 using the Method of Lagrange Multipliers.

Suppose we wish to find the maximum or minimum values of a function $f(x, y)$ for (x, y) restricted to a curve C in the plane. Then $\nabla f(a, b)$ is *perpendicular* to C at any max/min point $(a, b) \in C$. Now suppose C is given as a level curve of a function, $C = \{(x, y) \text{ with } g(x, y) = c\}$. Then $\nabla f(a, b)$ is *parallel* to the gradient $\nabla g(a, b)$.

Thus, to find max/min point candidates $(x, y) = (a, b)$, we solve the following system of equations for x, y, λ:

$$\begin{cases} g(x,y) = c \\ \nabla f(x,y) = \lambda \nabla g(x,y) \end{cases} \iff \begin{cases} g(x,y) = c \\ \dfrac{\partial f(x,y)}{\partial x} = \lambda \dfrac{\partial g(x,y)}{\partial x} \\ \dfrac{\partial f(x,y)}{\partial y} = \lambda \dfrac{\partial g(x,y)}{\partial y}. \end{cases}$$

For Prob 1, take $f(x, y) = xy$, $g(x, y) = x^2 + y^2$, and solve for x, y, λ.