Math 254H **Weekly Homework 2** **Fall 2022**

A *linear mapping* $L : \mathbb{R}^n \to \mathbb{R}^m$ is a mapping compatible with vector addition and scalar multiplication, meaning for any vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ and scalars $s, t \in \mathbb{R}$, we have:

$$L(s\mathbf{u} + t\mathbf{v}) \;=\; s\,L(\mathbf{u}) + t\,L(\mathbf{v}).$$

For $L : \mathbb{R}^2 \to \mathbb{R}^2$, this means L is specified by the two outputs $L(\mathbf{i}) = L(1,0) = (a,b)$ and $L(\mathbf{j}) = L(0,1) = (c,d)$. For a general vector $(x,y) = x\,(1,0) + y\,(0,1)$, we have:

$$L(x,y) \;=\; x\,(a,b) + y\,(c,d) \;=\; (ax{+}cy,\, bx{+}dy),$$

so that a, b, c, d are slope coefficients, which we write in a 2×2 matrix:

$$[\,L\,] \;=\; \left[\begin{array}{c|c} a & c \\ b & d \end{array}\right].$$

When computing with matrices, we usually write a vector $\mathbf{v} = (x,y)$ in column form as $[\mathbf{v}] = \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$. We define multiplication of the matrix $[L]$ times the vector $[\mathbf{v}]$ so that it produces the above output: $[L] \cdot [\mathbf{v}] \overset{\text{def}}{=} [L(\mathbf{v})]$:

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} \overset{\text{def}}{=} \begin{bmatrix} ax + cy \\ bx + dy \end{bmatrix} \;=\; \begin{bmatrix} (a,c)\cdot(x,y) \\ (b,d)\cdot(x,y) \end{bmatrix}.$$

Thus, matrix multiplication takes dot products of row vectors of $[L]$ with $\mathbf{v}$.

Problems

1. For two linear mappings $L_1, L_2 : \mathbb{R}^2 \to \mathbb{R}^2$, the composite function $L_3 = L_1 \circ L_2$ is defined by $L_3(\mathbf{v}) = L_1(L_2(\mathbf{v}))$. Prove L_3 is a linear mapping.

2. Given the matrices:

$$[\,L_1\,] \;=\; \left[\begin{array}{c|c} a_1 & c_1 \\ b_1 & d_1 \end{array}\right], \qquad [\,L_2\,] \;=\; \left[\begin{array}{c|c} a_2 & c_2 \\ b_2 & d_2 \end{array}\right],$$

compute $L_2(x,y)$ and input the result into $L_1(x,y)$ to find the composite $L_3(x,y) = L_1(L_2(x,y))$. Determine its slope coefficients to find the matrix $[L_3]$, which we define as the *matrix product*: $[L_1] \cdot [L_2] \overset{\text{def}}{=} [L_3]$.

As before, interpret each entry of $[L_3]$ as a dot product of certain row and column vectors of $[L_1]$ and $[L_2]$.

3. Write the matrix of a general plane rotation $R = \mathrm{Rot}_\theta$, counterclockwise by angle θ as well as an explicit formula for $R(x, y)$. *Hint:* Find $R(1, 0)$ and $(0, 1)$ by trigonometry, then compute $R(x, y) = R(\mathbf{v}) = [R] \cdot [\mathbf{v}]$.

4. Compute the matrix of the composite mapping $\mathrm{Rot}_\alpha \circ \mathrm{Rot}_\beta$, and interpret it geometrically as a known linear mapping.

5. Let $\mathbf{u}_\theta = (\cos(\theta), \sin(\theta))$ be the unit vector with angle θ from the x-axis. Let $\mathrm{Ref}_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be the reflection of the plane which flips $\mathbf{u}_\theta$ across the line perpendicular to $\mathbf{u}_\theta$. (This is called an *orthogonal reflection* since the flipped and fixed lines are perpendicluar.)

Problem: Find the matrix of Ref_θ.

Hint: Recall that $\mathrm{Ref}_\theta(\mathbf{v}) = \mathbf{v} - 2\mathbf{p}$, where $\mathbf{p}$ is the orthogonal projection of $\mathbf{v}$ onto the direction $\mathbf{u}_\theta$.

6. Find the matrix of the composite mapping $\mathrm{Ref}_\alpha \circ \mathrm{Ref}_\beta$, and interpret it geometrically as a known linear mapping.

7. Fix a unit vector $\mathbf{m} = (m_1, m_2)$ with $|\mathbf{m}| = 1$, and consider the mappings $\ell_1 : \mathbb{R} \to \mathbb{R}^2$ and $\ell_2 : \mathbb{R}^2 \to \mathbb{R}$ given by $\ell_1(t) = \mathbf{m}\, t$ and $\ell_2(\mathbf{v}) = \mathbf{m} \cdot \mathbf{v}$.
a. Compute a formula for the composite $\ell_3(\mathbf{v}) = \ell_1(\ell_2(\mathbf{v}))$.
b. Re-do this by multiplying matrices: $[\ell_3] = [\ell_1] \cdot [\ell_2]$.
c. Interpret $\ell_3 : \mathbb{R}^2 \to \mathbb{R}^2$ geometrically as a known linear mapping.
Hint: Picture the effect of ℓ_2 followed by ℓ_1, applied to an input vector $\mathbf{v}$.