1. The Conservative Vector Field Theorem is only the degree 1 case of a more general result, the Poincare Lemma for differential forms. Here we prove the degree 2 case.

Vector Potential Theorem. For a differentiable vector field $\mathbf{G} : \mathbb{R}^3 \to \mathbb{R}^3$, the following conditions are equivalent.

(i) *Vector potential:* $\mathbf{G} = \operatorname{curl} \mathbf{F}$ for some vector field $\mathbf{F}$.

(ii) *Surface independent.* The flux of $\mathbf{G}$ enclosed by a loop does not depend on the surface chosen across the loop:

$$\iint_{S_1} \mathbf{G} \cdot d\mathbf{n}_1 \;=\; \iint_{S_2} \mathbf{G} \cdot d\mathbf{n}_2 \,,$$

where S_1, S_2 are two oriented surfaces with the same boundary curve.[1]

(iii) *Flux free.* For any closed surface S:

$$\oiint_S \mathbf{G} \cdot d\mathbf{n} = 0.$$

(iv) *Incompressible:* $\operatorname{div} \mathbf{G} = 0$ everywhere.

PROBLEM: Prove the following implications in analogy with the corresponding proofs for the Conservative Vector Field Theorem. Explicitly quote previous theorems used.

$$\text{(i)} \Rightarrow \text{(ii)}, \qquad \text{(ii)} \Rightarrow \text{(iii)}, \qquad \text{(iii)} \Rightarrow \text{(iv)}, \qquad \text{(iv)} \Rightarrow \text{(iii)}.$$

I am *not* asking you to prove the remaining implications (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i).

2. Consider the electric field $\mathbf{E}$ of uniform charge along an axis defined by the unit vector ℓ, and its potential energy function ϕ:

$$\mathbf{E}(\mathbf{v}) = \frac{\mathbf{v}_{\perp\ell}}{|\mathbf{v}_{\perp\ell}|^2}, \qquad \phi(\mathbf{v}) = -\ln|\mathbf{v}_{\perp\ell}|.$$

Here $\mathbf{v}_{\perp\ell}$ is the component of $\mathbf{v}$ perpendicular to the unit vector ℓ.

a. Verify that $\nabla\phi = -\mathbf{E}$ using vector calculus identities below, as well as the projection formula for $\mathbf{v}_{\perp\ell}$. Remember $\ell \cdot \ell = 1$.

b. Show $\mathbf{E}$ can be obtained as the integral of point-charge fields along the axis with constant charge density $\frac{1}{2}$:

$$\mathbf{E}(\mathbf{v}) = \int_{t=-\infty}^{\infty} \frac{1}{2} \frac{\mathbf{v} - t\ell}{|\mathbf{v} - t\ell|^3}\, dt.$$

Hints: $\int_a^b f(t)\mathbf{v}\, dt = (\int_a^b f(t)\, dt)\, \mathbf{v}$ for a constant vector $\mathbf{v}$,
and $|\mathbf{v} - t\ell| = \sqrt{t^2 - 2(\mathbf{v}\cdot\ell)t + |\mathbf{v}|^2} = \sqrt{u^2 + c^2}$
for $u = t - (\mathbf{v}\cdot\ell)$ and $c^2 = |\mathbf{v}|^2 - (\mathbf{v}\cdot\ell)^2 = |\mathbf{v}_{\perp\ell}|^2 \geq 0$.

[1] *Orientation* means a choice of "upward" normal $\mathbf{n}$, inducing counter-clockwise boundary $\mathbf{c}(t)$.

3. Consider the magnetic field $\mathbf{B}$ defined with respect to a unit vector ℓ, and its vector potential $\mathbf{A}$:

$$\mathbf{B}(\mathbf{v}) = \ell \times \mathbf{v}, \qquad \mathbf{A}(\mathbf{v}) = -\tfrac{1}{2}|\mathbf{v}_{\perp\ell}|^2\,\ell.$$

a. Sketch $\mathbf{B}$, starting at a point on the axis $\mathbf{v} = t\ell$, then on circles $\mathbf{v} = t\ell + \mathbf{v}_{\perp\ell}$ with $|\mathbf{v}_{\perp\ell}| = 1, 2, \ldots$. Separately sketch $\mathbf{A}$, and explain visually why $\operatorname{curl}\mathbf{A} = \mathbf{B}$.

b. Show that $\operatorname{div}\mathbf{B} = 0$ everywhere.

c. Show that $\operatorname{curl}\mathbf{A} = \mathbf{B}$.

d. By Ampere's Law, what current field $\mathbf{J}$ would induce the static magnetic field $\mathbf{B}$?

Vector Calculus Identities for scalar functions $f(\mathbf{v}), g(\mathbf{v})$ and vector fields $\mathbf{F}(\mathbf{v}), \mathbf{G}(\mathbf{v})$.

- $\nabla(\ell \cdot \mathbf{v}) = \ell, \quad \nabla(|\mathbf{v}|^2) = 2\mathbf{v}$
- $\operatorname{curl}(\mathbf{v}) = \mathbf{0}, \quad \operatorname{div}(\mathbf{v}) = 3$
- $\nabla(f(g(\mathbf{v})) = f'(g(\mathbf{v}))\,\nabla g(\mathbf{v})$, for $f : \mathbb{R} \to \mathbb{R}$
- $\nabla(fg) = (\nabla f)\,g + f\,\nabla g$
- $\operatorname{div}(f\mathbf{G}) = \nabla \cdot (f\mathbf{G}) = (\nabla f) \cdot \mathbf{G} + f\,\nabla\cdot\mathbf{G}$
- $\operatorname{div}(\mathbf{F} \times \mathbf{G}) = \nabla \cdot (\mathbf{F} \times \mathbf{G}) = (\nabla \times \mathbf{F}) \cdot \mathbf{G} - \mathbf{F} \cdot (\nabla \times \mathbf{G})$
- $\operatorname{curl}(f\mathbf{G}) = \nabla \times (f\mathbf{G}) = \nabla f \times \mathbf{G} + f\,\nabla\times\mathbf{G}$