**Math 254H** # **Weekly Homework 6** **Fall 2022**

**1.** Area bounded by a curve

**a.** Let $\vec{c}(t) = (x(t), y(t))$ for $t \in [a, b]$ be a parametric curve moving left-to-right above the $x$-axis, so that $x'(t) \geq 0$ and $y(t) \geq 0$. In MTH 133, we learn that the area under $\vec{c}(t)$ and above the $x$-axis is $\int y\,dx = \int_a^b y(t)x'(t)\,dt$.

Prove this formula by parametrizing the region and using the Substitution Theorem for double integrals.

**b.** How does the formla $\int y\,dx$ behave if $y < 0$ or $x' < 0$ always or sometimes? What geometric quantities does it compute in the different cases?

**c.** Now let $\vec{c}(t)$ be a counterclockwise loop bounding a region $D$, with the lower half of $\vec{c}(t)$ moving left-to-right, the upper half returning right-to-left. Find a single-integral formula similar to $\int y\,dx$ which computes the area of $D$.

**d.** Give an example of a vector field $\mathbf{F}(x, y)$ with curl $\mathbf{F}(x, y) = 1$ everywhere. Apply the Curl Theorem to $\mathbf{F}$, $D$, and $\vec{c}$ to give a formula similar to part (c), a single integral in $x(t), y(t)$ over $t \in [a, b]$ that computes enclosed area. Also, find an appropriate $\mathbf{F}$ that proves your exact formula from part (c).

**2.** In Problem Sheet 16-9 below, do #7, 9, 12. The idea is to take consider $u = f(x, y)$ and $v = g(x, y)$ as functions whose contour lines give a parametric coordinate grid on the $xy$-plane. To invert these into a parametrization $G(u, v) = (x, y)$, solve the equations $u = f(x, y), v = g(x, y)$ for $x, y$.

In each of Problems 1–6, solve for $x$ and $y$ in terms of $u$ and $v$ and then compute the Jacobian $\partial(x, y)/\partial(u, v)$.

**1** $u = x + y, \quad v = x - y$
**2** $u = x - 2y, \quad v = 3x + y$
**3** $u = xy, \quad v = \dfrac{y}{x}$
**4** $u = 2(x^2 + y^2), \quad v = 2(x^2 - y^2)$
**5** $u = x + 2y^2, \quad v = x - 2y^2$
**6** $u = \dfrac{2x}{x^2 + y^2}, \quad v = \dfrac{-2y}{x^2 + y^2}$

**7** Let $R$ be the parallelogram bounded by the lines $x + y = 1$, $x + y = 2$ and $2x - 3y = 2$, $2x - 3y = 5$. Substitute $u = x + y$, $v = 2x - 3y$ to find the area $A = \iint_R dx\,dy$ of $R$.
**8** Substitute $u = xy$, $v = y/x$ to find the area of the first quadrant region bounded by the lines $y = x$, $y = 2x$ and the hyperbolas $xy = 1$, $xy = 2$.
**9** Substitute $u = xy$, $v = xy^3$ to find the area of the region in the first quadrant bounded by the curves $xy = 2$, $xy = 4$ and $xy^3 = 3$, $xy^3 = 6$.
**10** Find the area of the region in the first quadrant bounded by the curves $y = x^2$, $y = 2x^2$ and $x = y^2$, $x = 4y^2$. (*Suggestion:* Let $y = ux^2$ and $x = vy^2$.)
**11** Use the method of Problem 10 to find the area of the region in the first quadrant bounded by the curves $y = x^3$, $y = 2x^3$ and $x = y^3$, $x = 4y^3$.
**12** Let $R$ be the region in the first quadrant bounded by the circles $x^2 + y^2 = 2x$, $x^2 + y^2 = 6x$ and the circles $x^2 + y^2 = 2y$, $x^2 + y^2 = 8y$. Use the transformation $u = 2x/(x^2 + y^2)$, $v = 2y/(x^2 + y^2)$ to evaluate the integral $\iint_R (x^2 + y^2)^{-2}\,dx\,dy$.
**13** Use elliptical coordinates $x = 3r\cos\theta$, $y = 2r\sin\theta$ to find the volume of the region that is bounded by the $xy$-plane, the paraboloid $z = x^2 + y^2$, and the elliptical cylinder $x^2/9 + y^2/4 = 1$.
**14** Let $R$ be the solid ellipsoid with outer boundary surface $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$. Use the transformation $x = au$, $y = bv$, $z = cw$ to show that the volume of this ellipsoid is $V = \iiint_R 1\,dx\,dy\,dz = \frac{4}{3}\pi abc$.
**15** Find the volume of the region in the first octant that is bounded by the hyperbolic cylinders $xy = 1$, $xy = 4$; $xz = 1$, $xz = 9$; $yz = 4$, $yz = 9$. (*Suggestion:* Let $u = xy$, $v = xz$, $w = yz$, and note that $uvw = x^2y^2z^2$.)
**16** Use the transformation $x = (r/t)\cos\theta$, $y = (r/t)\sin\theta$, $z = r^2$ to find the volume of the region $R$ that lies between the paraboloids $z = x^2 + y^2$, $z = 4(x^2 + y^2)$ and also between the planes $z = 1$, $z = 4$.
**17** Let $R$ be the rotated elliptical region bounded by the graph of $x^2 + xy + y^2 = 3$. Let $x = u + v$ and $y = u - v$. Show that

$$\iint_R e^{-(x^2+xy+y^2)}\,dx\,dy = 2\iint_S e^{-(3u^2+v^2)}\,du\,dv.$$

Then substitute $u = r\cos\theta$, $v = \sqrt{3}(r\sin\theta)$ to evaluate the latter integral.
**18** Derive Relation (6) between the Jacobians of a transformation and its inverse from the chain rule and the following property of determinants:

$$\begin{vmatrix} a_1 & b_1 \\ c_1 & d_1 \end{vmatrix} \cdot \begin{vmatrix} a_2 & b_2 \\ c_2 & d_2 \end{vmatrix} = \begin{vmatrix} a_1a_2 + b_1c_2 & a_1b_2 + b_1d_2 \\ a_2c_1 + c_2d_1 & b_2c_1 + d_1d_2 \end{vmatrix}.$$