Math 254H **Recitation 1** **Sept 1, 2022**

Prove the following propositions of plane geometry by translating them into vector algebra. Do not use trigonometry or rely on known geometric results, only basic definitions.

1. The centroid of a triangle lies on each median, $\frac{2}{3}$ of the way from each vertex to the midpoint of the opposite side. (Thus, the centroid is the intersection point of the three medians.)

2. Given a triangle, the midpoints of its sides are by definition the vertices of the *medial triangle*. Show that the medial triangle has the same centroid as the original triangle.

3. The medial triangle is similar to the original triangle, with sides parallel to and half the length of the original sides.

4. Let $ABCD$ be a pallelogram with $AB||CD$ and $AC||BD$. Then $ABCD$ is a rhombus (having all side-lengths equal) if and only if the diagonals AC and BD are perpendicular to each other. *Hint:* Work with the side vectors $\vec{u} = \vec{AB} = \vec{CD}$, $\vec{v} = \vec{AC} = \vec{BD}$, not with vectors for the individual vertices.

5. (a) If a circle with center O has an inscribed segment AB with midpoint M, then AB is perpendicular to OM. *Hint:* Take the origin to be the center O.
(b) If a triangle ABC is inscribed in the circle, then the orthogonal bisectors of the three sides meet at O. *Hint:* The radial line OM is orthogonal to AB, so it is is the orthogonal bisector.

6. (a) If an angle is inscribed in a circle with one side being a diameter, the corresponding central angle is twice as large as the inscribed angle. *Hint:* Show that the inscribed angle is equal to the central angle between the diameter and the midpoint of the secant cutting across the inscribed angle.
(b) Any inscribed angle is half the size of the corresponding central angle.

7. If a circle has inscribed segments AB and CD intersecting at P, then:
$$|AP| \cdot |BP| = |CP| \cdot |DP| .$$
Hint: Start with the intersection point $P = \vec{p}$, and let $A = \vec{p} + \vec{u}$, $B = \vec{p} - t\vec{u}$, $C = \vec{p} + \vec{v}$, $D = \vec{p} - s\vec{v}$, satisfying $|A|^2 = |B|^2 = |C|^2 = |D|^2 = r$. Solve for t, s in terms of $\vec{p}, \vec{u}, \vec{v}$, and evaluate $t|\vec{u}|^2$, $s|\vec{v}|^2$.

8. (a) For a circle with center O, a line tangent at M is perpendicular to radius OM. *Hint:* A tangent line $\vec{\ell}(t) = \vec{m} + t\vec{v}$ touches the circle only at $\vec{m}$.
(b) From a point P outside the circle, if the two tangent lines touch the circle at Q and R, then $|PQ| = |PR|$.
(c) FOR FUN: Dandelin spheres demonstrate that a conic section is an ellipse, using equality of tangents.