Math 254H 4/20/2020 Review

Function $P: \mathbb{R}^k \longrightarrow \mathbb{R}^n$

$$P(u_1, \dots, u_k) = (x_1(u_1 \cdots u_k), \dots, x_n(u_1, \dots, u_k))$$

picture: k-dimensional "surface" in $\mathbb{R}^n$

$k=n \Longrightarrow$ coordinate system

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matrix represents linear function.

tangent vectors of u-grid

$$\text{Jacobian } \underset{\substack{n \times k \\ \text{row col}}}{[DP_{\vec u}]} = \begin{bmatrix} \nabla x_1(u_1 \cdots u_n) \\ \vdots \\ \nabla x_n(u_1 \cdots u_k) \end{bmatrix} = \left[\frac{\partial P}{\partial u_1} \Big| \cdots \Big| \frac{\partial P}{\partial u_k} \right] \updownarrow n$$

$P: \mathbb{R}^1 \longrightarrow \mathbb{R}^3$

parametric curve

$\vec c(t) = P(t)$ $\quad \frac{\partial P}{\partial t} = \vec c\,'(t)$

$P: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

parametric surface

$P: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

parametrized region (coordinate system)

Linear approx: $P(\vec u + \vec h) \approx P(\vec u) + DP_{\vec u}(h)$

"field" of linear functions: DP

each $\vec u \in \mathbb{R}^k \Longrightarrow \boxed{DP_{\vec u}: \mathbb{R}^k \to \mathbb{R}^n}$ linear fun

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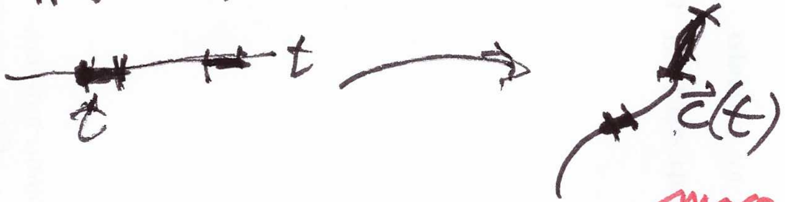
stretching factor: near $\vec u \in \mathbb{R}^k$ compare $vol_k(S^*)$ vs $vol_k(S) = vol\, P(S^*)$

$P: S^* \longrightarrow S$

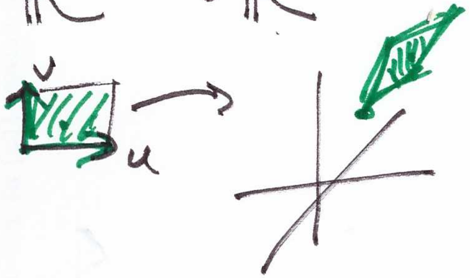
$P: \mathbb{R} \to \mathbb{R}^3$ stretch $= |\vec{c}\,'(t)|$ speed!



Subs formula:

$$\text{mass} = \int_{\vec c} \underset{\text{density fun}}{f(\vec c)}\, dc = \int_t \underset{\text{density}}{f(\vec c(t))}\ \overset{\text{stretch factor}}{\underbrace{|\vec c\,'(t)|\,dt}_{\Delta c}}$$

$P: \mathbb{R}^2 \to \mathbb{R}^3$



$$\text{stretch} = \left|\frac{\partial P}{\partial u} \times \frac{\partial P}{\partial v}\right| \overset{!}{=} \sqrt{\det \underbrace{\begin{pmatrix} \frac{\partial P}{\partial u}\cdot\frac{\partial P}{\partial u} & \frac{\partial P}{\partial u}\cdot\frac{\partial P}{\partial v} \\ \frac{\partial P}{\partial v}\cdot\frac{\partial P}{\partial u} & \frac{\partial P}{\partial v}\cdot\frac{\partial P}{\partial v} \end{pmatrix}}_{\text{Gramian matrix}}}$$

subs: $\iint_S f\, dS = \iint_{S^*} f(P(u,v)) \cdot (\text{stretch})\, du\, dv$

$P: \mathbb{R}^3 \to \mathbb{R}^3$ stretch $= |\det[DP]| = \sqrt{\det [DP]^T \det [DP]}$

$\det DP < 0$ means
P is "mirror image"

$\det[DP]^T = \det[DP]$

subs formula

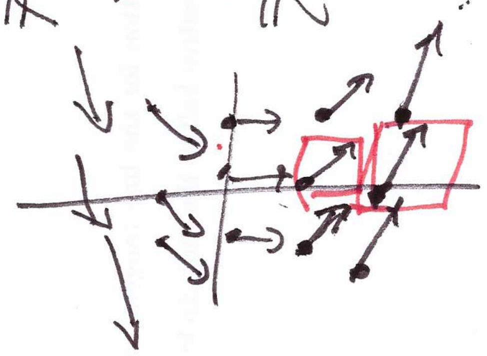
Fields: near each point of $\mathbb{R}^n$,

get a local object: geometric, a function...

each object is in its own space, near pt.

Vector field $\vec{F}: \mathbb{R}^n \longrightarrow \mathbb{R}^n$

$F(x,y) = (1, x)$



e.g. force field: force vector at each point.

gradient: $\nabla f = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right)$ dir of max increase

$f: \mathbb{R}^n \to \mathbb{R}$ $\quad f(x_1, \dots, x_n)$

One-form: covector field $\quad \boxed{df_{\vec{x}} = Df_{\vec{x}}} : \mathbb{R}^n \to \mathbb{R}$

$f(\vec{x}+\vec{h}) \approx f(\vec{x}) + Df_{\vec{x}}(\vec{h}) \qquad dx_i(\vec{h}) = dx_i(h_1, \dots, h_n) = h_i$

"differential of f" $\quad df = \frac{\partial f}{\partial x_1} dx_1 + \dots + \frac{\partial f}{\partial x_n} dx_n$

$df_{\vec{x}} = \frac{\partial f}{\partial x_1}(\vec{x})\, dx_1 + \dots + \frac{\partial f}{\partial x_n}(\vec{x})\, dx_n$

$df_x(\vec{h}) = \nabla f(\vec{x}) \cdot \vec{h}$

$df = (\nabla f)^*$ dual

Scalar field ~~$\mathbb{R}$,~~

$f : \mathbb{R}^n \to \mathbb{R}$ at each $\vec{x} \in \mathbb{R}^n$, $f(\vec{x})$ = scalar

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Fields are measured by integrals

"0-form" Scalar field f $\quad \int_R f \, dR$ = "total mass" if f = density.

Vector field $\vec{F}$ $\quad \int \vec{F} \cdot d\vec{c}$ = line integral, total pull of force $\vec{F}$ along $\vec{c}$

$\updownarrow$ equivalent

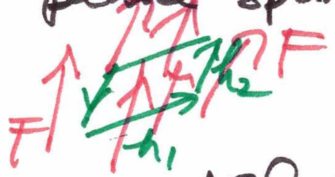
One-form φ $\quad \int_{\vec{c}} \varphi = \int_t \varphi(\vec{c}\,'(t)) \, dt$

Field, integral measure $\Longrightarrow$ $\left(\begin{array}{c}\text{derivative} \\ = \text{rate of integral} \\ \text{near each } \vec{x}\end{array}\right)$ new field

$\vec{F}$, vector field, line integral circulation $\Longrightarrow$ rate of circulation (curl $\vec{F}$)($\vec{x}$) "vector field"

$(\underbrace{\text{curl}_{yz}\vec{F}}, -\underbrace{\text{curl}_{xz}\vec{F}}, \underbrace{\text{curl}_{xy}\vec{F}})$

1-form φ, line integral $\Longrightarrow$ rate of circulation (near $\vec{x}$)
circulation in plane spanned by $\vec{h}_1, \vec{h}_2$



$\varphi_x : \mathbb{R}^n \to \mathbb{R}$

2-form $d\varphi_{\vec{x}}(\vec{h}_1, \vec{h}_2)$

$\eta = d\varphi_x : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$

2-form η, surface integral $\Longrightarrow$ rate of flux

$$\iint_S \eta = \iint_{S^*} \eta\left(\frac{\partial P}{\partial u}, \frac{\partial P}{\partial v}\right) du\,dv \qquad \omega = d\eta \text{ 3-form}$$

flux integral

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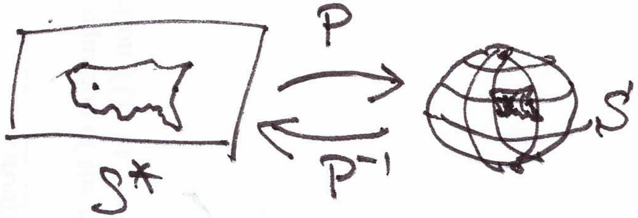
Two kinds of functions

⊙ Mappings

$$\begin{array}{rccc} P: & \mathbb{R}^k & \longrightarrow & \mathbb{R}^n \\ & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \text{take:} & pt & \longrightarrow & pt \\ & S^* & \longrightarrow & S \end{array}$$

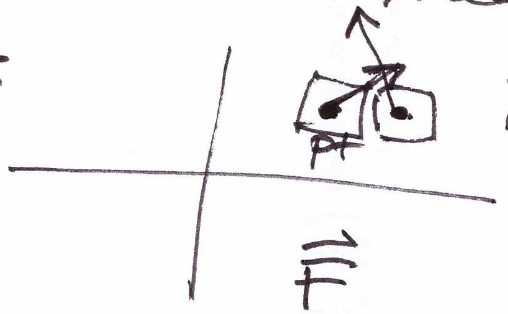
$P(u_1,\dots,u_k) = (x_1,\dots,x_n)$

P parametrizes output set S



$$\int_S f\,dS = \int_{S^*} f(P(u))\underline{(\text{stretch})}\,du$$

⊙ Fields: $\vec{F}$, φ, ω, ... give an "infinitesimal object"
on $\mathbb{R}^n$ near each point $\vec{x} \in \mathbb{R}^n$
points



$\mathbb{R}^n = \mathbb{R}^2$

differential
k-form φ
at $\vec{x}$, $\varphi_x(\vec{h}_1,\dots,\vec{h}_k) = $ scalar

operations on diff forms

wedge prod. $\underset{k\text{-form}}{\varphi} \wedge \underset{l\text{-form}}{\psi} = (k+l)$ form

exterior derivative

$\varphi = \sum f_{..}\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}$

$d\varphi = \sum df \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}$

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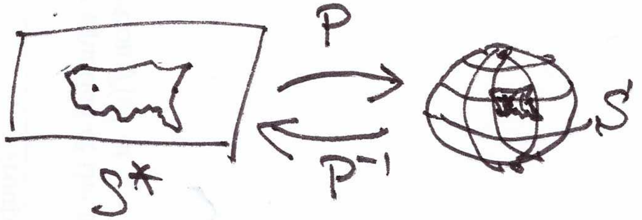
Two kinds of functions

⊙ Mappings $P: \mathbb{R}^k \to \mathbb{R}^n$ $P(u_1, \dots, u_k) = (x_1, \dots, x_n)$

$$\begin{array}{rccc} & \mathbb{R}^k & \longrightarrow & \mathbb{R}^n \\ & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \text{take:} & pt & \longrightarrow & pt \\ & S^* & \longrightarrow & S \end{array}$$

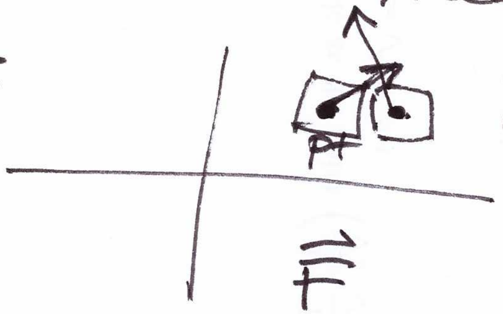
P parametrizes output set S



$$\int_S f \, dS = \int_{S^*} f(P(u)) (\text{stretch}) \, du$$

⊙ Fields: $\vec{F}$, φ, ω, … give an "infinitesimal object" near each point $\vec{x} \in \mathbb{R}^n$

on $\mathbb{R}^n$ points



$\mathbb{R}^n = \mathbb{R}^2$

differential k-form φ

at $\vec{x}$, $\varphi_x(\vec{h}_1, \dots, \vec{h}_k) = \text{scalar}$

operations on diff forms

wedge prod. $\underset{k\text{-form}}{\varphi} \wedge \underset{l\text{-form}}{\psi} = (k+l)$ form

exterior derivative

$\varphi = \sum f_{..} \, dx_{i_1} \wedge \dots \wedge dx_{i_k}$

$d\varphi = \sum df \wedge dx_{i_1} \wedge \dots \wedge dx_{i_k}$

exterior derivative ⟷ vector field derivatives

$f \xrightarrow{d} \varphi = df$ $\qquad\qquad$ $f \xrightarrow{grad} \nabla f = \vec{F}$

fun
0-form $\qquad$ 1-form

$$\varphi = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz \qquad\qquad \nabla f = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right)$$

$\varphi \xrightarrow{d} d\varphi = \eta$ $\qquad\qquad$ $\vec{F} \xrightarrow{curl} \operatorname{curl} \vec{F}$

1-form $\qquad$ 2-form $\qquad$ (p, q, r) $\qquad$ $(\tilde{p}, \tilde{q}, \tilde{r})$

$p\,dx + q\,dy + r\,dz$

$\tilde{p}\, dy \wedge dz - \tilde{q}\, dx \wedge dz + \tilde{r}\, dx \wedge dy$ $\qquad$ $p\vec{i} + q\vec{j} + r\vec{k}$

$\eta \xrightarrow{d} \omega = d\eta$ $\quad g\, dx \wedge dy \wedge dz$ $\qquad\qquad$ $\vec{F} \xrightarrow{div} \operatorname{div} \vec{F}$

2-form $\qquad$ 3-form $\qquad\qquad\qquad$ g

Stokes Thm: $\displaystyle\int_{\partial S} \varphi = \int_S d\varphi$ ⟷ Grad Thm
Curl Thm
Div Thm

∂S = boundary of S

Conservative Vector Field Theorem: 4 Equivalent Conditions.

1st FTC ⇑ ⇓ Fund Thm

- ⊙ $\vec{F}$ is conservative: $\vec{F} = \nabla f$, some f
- ⊙ Path-indep: $\int \vec{F} \cdot d\vec{c}_1 = \int \vec{F} \cdot d\vec{c}_2$ if $\vec{c}_1(0) = \vec{c}_2(0)$, $\vec{c}_1(1) = \vec{c}_2(1)$

⇕ elementary

- ⊙ Circulation-free: $\oint \vec{F} \cdot d\vec{c} = 0$, $\vec{c}$ closed, $\vec{c}(0) = \vec{c}(1)$

curl ⇑ ⇓ Geom def of curl = rate of circulation.

- ⊙ Irrotational: $\text{curl}\,\vec{F} = \vec{0}$ everywhere.

Generalization: $\overset{k\text{-form}}{\alpha} \longrightarrow \overset{(k+1)\text{-form}}{d\alpha} = \eta$

When does a $(k+1)$-form η have potential α?

$\eta = d\alpha$

⇕

when η has $d(\eta) = 0$

"Poincare Lemma" Easy to prove!

$f \to \nabla f = \vec{F}$

$\vec{F} \to \text{curl}\,\vec{F} = \vec{G}$

$\vec{G} \downarrow$

$\text{div}\,\vec{G}$

Given $\vec{G}$ vect field (dual to 2-form)

when is $\vec{G} = \text{curl}\,\vec{F}$? (deriv of 1-form)

Ans: when $\text{div}\,\vec{G} = 0$

$\vec{F}$ called "vector potential"

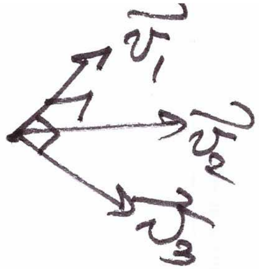
"magnetic ~~field~~ potential"

Maxwell 2: $\text{div}\,\vec{B} = 0$

$\Longrightarrow$ exists $\vec{A}$ with $\text{curl}\,\vec{A} = \vec{B}$

$\left.\begin{array}{l}\underline{\text{Computations}}\\ \underline{\text{Pictures}}\end{array}\right\}$ Quizzes, WHW

Find area of cone
(right circular cone)



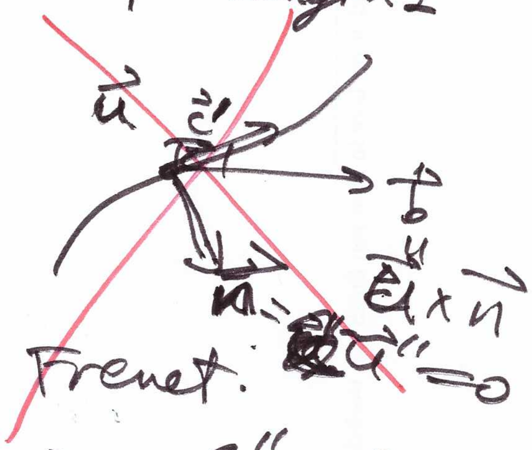
1. Parametrize: appropriate coord vectors. (orthonormal)
perp length 1

$\vec{v}_1 = \frac{(1,2,3)}{\sqrt{1^2+2^2+3^2}}$, $\vec{v}_2 = \frac{(-2,1,0)}{\sqrt{2^2+1^2}}$

perp

$\vec{v}_1 \cdot \vec{v}_2 = 0$

$\vec{v}_3 = \underline{\vec{v}_1 \times \vec{v}_2}$



$c''=0$

$P(u, v) = u\vec{v}_1 + \cos(v)\vec{v}_2 + \sin(v)\vec{v}_3$

$(u,v) \in S^*$

const u

stretch

2. Surface area: $\iint_{S} 1\,dS = \iint_{S^*} 1 \left|\frac{\partial P}{\partial u} \times \frac{\partial P}{\partial v}\right| du\,dv$