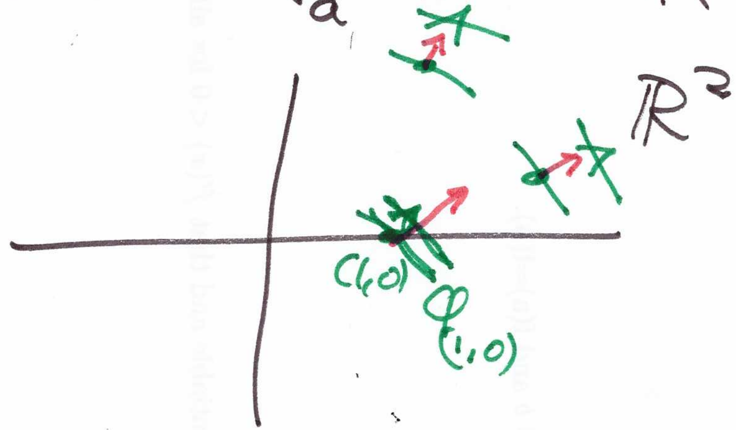
Math 254H 4/15/2020 Differential forms on $\mathbb{R}^3$ of degree 1

Diff form φ (phi) a covector field:

linear mapping

at each $\vec{a} \in \mathbb{R}^3$ gives a covector $\varphi_{\vec{a}} : \mathbb{R}^3 \to \mathbb{R}$



$$\left(\begin{array}{l} \text{covector } \lambda : \mathbb{R}^3 \to \mathbb{R} \\ \lambda(\vec{h}) = \vec{v} \cdot \vec{h} \\ \lambda = \vec{v}^* \quad \text{some } \vec{v} \in \mathbb{R}^3 \end{array} \right)$$

nonlinear

Derivative $f : \mathbb{R}^3 \to \mathbb{R}$

naturally produces a diff form $df = Df$

At each $\vec{a} \in \mathbb{R}^3$, $df_{\vec{a}} = Df_{\vec{a}} : \mathbb{R}^3 \to \mathbb{R}$ linear

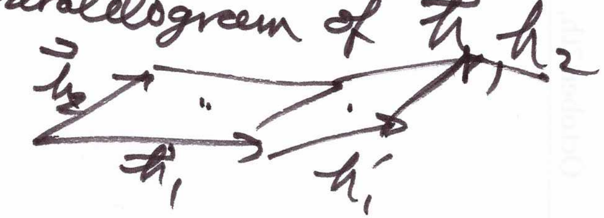
$$f(\vec{a} + \vec{h}) \approx f(\vec{a}) + \underline{df_a(\vec{h})} = f(\vec{a}) + \nabla f(\vec{a}) \cdot \vec{h}$$

extra step

$$df_{\vec{a}} = \nabla f(\vec{a})^*$$

derivative diff form = covector field of gradient field

Properties of $d\alpha_{\vec{a}}(\vec{h}_1, \vec{h}_2)$ = directional rate of circ with parallelogram of $\vec{h}_1, \vec{h}_2$



Linear in <u>each</u> variable!

"Bilinear" $d\alpha_{\vec{a}}(\vec{h}_1 + \vec{h}_1', \vec{h}_2) = d\alpha_{\vec{a}}(h_1, h_2) + d\alpha_{\vec{a}}(\vec{h}_1', \vec{h}_2)$

$d\alpha_{\vec{a}}(\vec{h}_1, \vec{h}_2 + \vec{h}_2') = \cdots + \cdots$

"Alternating". $d\alpha_{\vec{a}}(\vec{h}_2, \vec{h}_1) = -d\alpha_{\vec{a}}(\vec{h}_1, \vec{h}_2)$

[like $\det(\vec{h}_1, \vec{h}_2, \vec{h}_3)$
3×3 mat : $\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3$]

Define 2-form (degree 2 diff form)

$\eta_{\vec{a}} : \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}$ bilinear, alternating

eta $\quad (\vec{h}_1, \vec{h}_2)$ $\qquad \vec{\eta}_a(\vec{h}_2, \vec{h}_1) = -\vec{\eta}_a(\vec{h}_1, \vec{h}_2) \ldots$

Diff forms of degree 2.

0-form $\;\; f \longrightarrow df \longrightarrow d(df) = 0$ 2-form; 1-form

$\varphi \longrightarrow d\varphi$

Use use degree 1 diff forms to define

line integrals: $\displaystyle\int_{\vec c} df \overset{\text{def}}{=} \int \frac{\partial f}{\partial x} x'(t) + \frac{\partial f}{\partial y} y'(t) + \frac{\partial f}{\partial z} z'(t)\, dt$

$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$

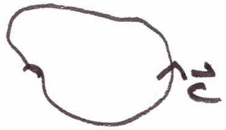
$\vec c = (x, y, z)(t)$

$(0 \le t \le 1)$

$= f(\vec c(1)) - f(\vec c(0))$

(Gradient Thm)

If $\vec c$ closed curve



$\oint_{\vec c} df = 0$ or $\oint_{\vec c} \varphi \neq 0\,?$ ← not a derivative df

"circulation" of φ

directional curls around each axis

~~curl~~ of vect field = vector

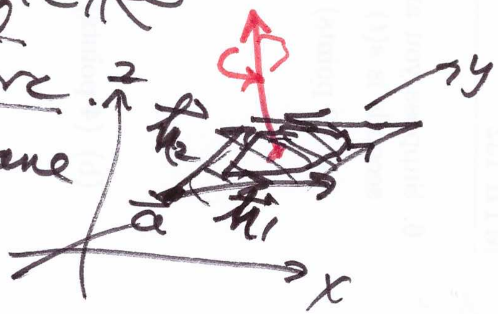
Natural derivative: rate of circulation → derivative of diff form φ

$d\varphi$ new object: what does it do?

at each point $\vec a \in \mathbb{R}^3$

~~Area~~ $d\varphi_{\vec a}$ measure $\lim_{\vec c \to \vec a} \dfrac{\oint_{\vec c} \varphi}{\text{area enclosed inside } \vec c}$ directional rate of circ. within plane

$d\varphi_{\vec a}(\underline{\vec h_1}, \underline{\vec h_2})$ ⟶



Describe algebraically: $\overset{\text{1-form}}{dx}(\vec{h}) = dx(h_1, h_2, h_3) = h_1$

$dy(h_1\ h_2\ h_3) = h_2$

$dz\ (h_1\ h_2\ h_3) = h_3$

Wedge product, alternating product

$$(dx \wedge dy)(\vec{h}_1, \vec{h}_2) = dx(\vec{h}_1) \overset{\text{multiply numbers}}{\longleftrightarrow} dy(\vec{h}_2)$$

$$- dy(\vec{h}_1)\ dx(\vec{h}_2)$$

$$= dx(\vec{h}_1)\, dy(\vec{h}_2) - dx(\vec{h}_2)\, dy(\vec{h}_1)$$

~~Basis of 2-forms~~: $(dx \wedge dx)(\vec{h}_1, \vec{h}_2)$

$$= dx(\vec{h}_1)\, dx(\vec{h}_2) - dx(\vec{h}_2)\, dx(\vec{h}_1) = 0$$

$$\left.\begin{matrix} dx \wedge dx \\ dy \wedge dy \\ \vdots \end{matrix}\right\} = 0$$

Bases of 2-forms

$dx \wedge dy$, $dx \wedge dz$, $dy \wedge dz$

neg

$\cancel{dy \wedge dx}$ $\quad \underset{0}{\cancel{dx \wedge dx}}$

Math 254 4/16/2020 Derivative of Differential Forms

0-forms $f: \mathbb{R}^3 \to \mathbb{R}$ function, at $\vec{a}$ get $f(\vec{a})$ number.

d

1-forms φ (phi) $\Rightarrow$ each $\vec{a} \in \mathbb{R}^3$ $\varphi_{\vec{a}}: \mathbb{R}^3 \to \mathbb{R}$ linear (covector)

covector field

$\varphi_{\vec{a}}(\vec{h})$ 1 input $\vec{h} \in \mathbb{R}^3$

d

2-forms η (eta) $\Rightarrow$ each $\vec{a} \in \mathbb{R}^3$ $\eta_{\vec{a}}: \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$

$\eta_{\vec{a}}(\vec{h}_1, \vec{h}_2)$ bilinear (linear in $\vec{h}_1$ and $\vec{h}_2$)

2 inputs $\vec{h}_1, \vec{h}_2 \in \mathbb{R}^3$

alternating

$\eta_{\vec{a}}(\vec{h}_1, \vec{h}_2) = -\eta_{\vec{a}}(\vec{h}_2, \vec{h}_1)$

d

3-forms ω (omega) $\Rightarrow \omega_{\vec{a}}(\vec{h}_1, \vec{h}_2, \vec{h}_3) = f(\vec{a}) \det(\vec{h}_1, \vec{h}_2, \vec{h}_3)$

linear in each $\vec{h}_i$

alternating: $\omega_{\vec{a}}(\vec{h}_1, \vec{h}_2, \vec{h}_3) = -\omega_{\vec{a}}(\vec{h}_2, \vec{h}_1, \vec{h}_3)$

$\omega_{\vec{a}}: \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ $= -\omega_{\vec{a}}(\vec{h}_1, \vec{h}_3, \vec{h}_2)$

$d\varphi$ = "exterior derivative" of form φ

directional rate of integral of φ

Unifying grad, curl div = exterior deriv

0-form f : "integral" = value $f(\vec{a})$

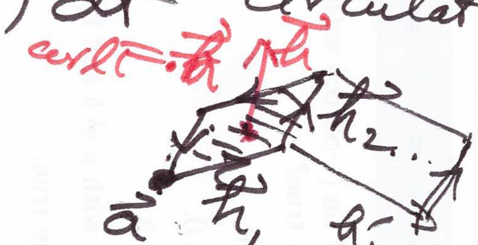
"grad" d ↓ rate of change of $f(\vec{a})$ in direction $\vec{h}$

$$= df_{\vec{a}}(\vec{h}) = \alpha \quad \text{1-form}$$

$$f(\vec{a}+\vec{h}) \simeq f(\vec{a}) + df_{\vec{a}}(\vec{h}) = f(\vec{a}) + \underline{\nabla f(\vec{a})}\cdot \underline{\vec{h}}$$

1-form α (not df for any f)

"integral" $\displaystyle\oint_{\vec{c}} \alpha \overset{def}{=} \int_{t=0}^{1} \alpha(\vec{c}\,'(t))\,dt$ circulation



d ↓ (rate of circulation at $\vec{a}$

"curl"

$$d\alpha_{\vec{a}}(\underbrace{\vec{h}_1,\vec{h}_2}_{\text{direction}}) = \lim_{\vec{c}\to\vec{a}} \frac{\int_{\vec{c}} \alpha}{\text{area enclosed}}$$

2-form

$\vec{c}$

$d\alpha_{\vec{a}}(\vec{h}_1, \vec{h}_2)$ bilinear

$\vec{c}(\vec{h}_1, \vec{h}_2) = -\vec{c}(\vec{h}_2, \vec{h}_1)$ reverse curve

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2-form η "flux" integral surface $\displaystyle\iint_S \eta = \iint \eta\left(\frac{\partial P}{\partial u}, \frac{\partial P}{\partial v}\right) du\,dv$

$S = P(u,v)$

"div" d ↓

$d\eta$ = rate of flux out of box $\quad d\eta_{\vec{a}}(\vec{h}_1, \vec{h}_2, \vec{h}_3)$

3-form

<u>Lemma:</u> φ 1-form, assume $d\varphi = 0$ "circulation-free"

$f(\vec{r}) = f(x,y,z)$ function

Then $d(f\varphi) = \underbrace{df}_{\text{1-form}} \wedge \underbrace{\varphi}_{\text{1-form}}$ 2-form

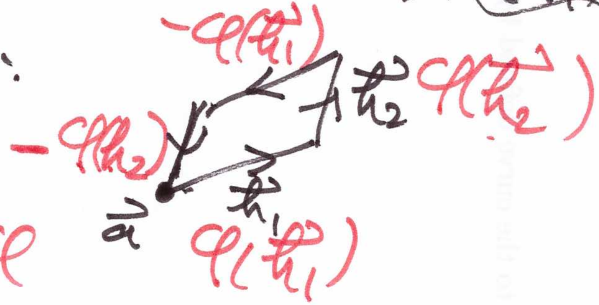
$\wedge$ ← wedge product, exterior product

$$(df \wedge \varphi)(\vec{h}_1, \vec{h}_2) = df(\vec{h}_1)\,\varphi(\vec{h}_2) - df(\vec{h}_2)\,\varphi(\vec{h}_1)$$

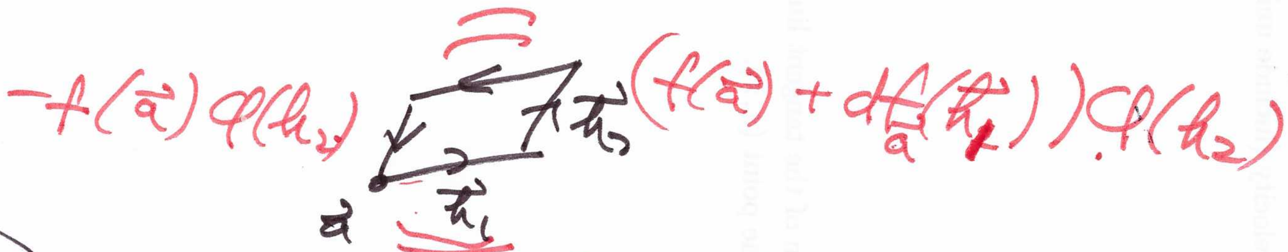
<u>Justification:</u>

$d\varphi = 0$

rate of circ of φ



$d(f\varphi)$



$$\lim_{h_1, h_2 \to 0} \frac{\oint f\varphi}{\text{area}} = df_{\vec{a}}(\vec{h}_1)\,\varphi(\vec{h}_2) - df_{\vec{a}}(\vec{h}_2)\,\varphi(\vec{h}_1)$$

$\varphi = p\,dx + q\,dy + r\,dz$

$d\varphi = dp \wedge dx + dq \wedge dy + dr \wedge dz$

$dx_{\vec{a}}(\vec{h}) = dx_{\vec{a}}(h_x, h_y, h_z) = h_x$

$d(df) = 0$

$df \wedge df = 0$

$$= \left(\frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy + \frac{\partial p}{\partial z}dz\right)\wedge dx + \left(\frac{\partial q}{\partial x}dx + \frac{\partial q}{\partial y}dy + \frac{\partial q}{\partial z}\right)\wedge dy$$

$$+ \left(\frac{\partial r}{\partial x} \quad \cdot \quad \cdot\right)\wedge dz$$

$$= \frac{\partial p}{\partial y}dy\wedge dx + \frac{\partial p}{\partial z}dz\wedge dx + \cdots\cdots$$

$$= \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\right)dx\wedge dy + (\qquad)dx\wedge dz + (\quad)dy\wedge dz$$

$$\text{curl } \vec{F} = \text{curl }(p, q, r)$$

<u>Fund Thm of Calc:</u> tot $=\int$ rate

Grad Thm: $f(\vec{c}(1)) - f(\vec{c}(0)) = \int_{\vec{c}} df$

Curl Thm: $\oint_{\vec{c}} \varphi = \iint_{S} d\varphi$

$\vec{c}$ = boundary of S

Div Thm: $\iint_{S} \eta = \iiint_{R} d\eta$

S = boundary of R

Stokes Thm for Differential Forms. (any $\mathbb{R}^n$)

Similarly for $d(\text{2-form}) = \text{3-form}$

$\xrightarrow{\text{div}}$