Math 254H 3/30/2020 Proof of Integral Thms

Multivariable functions $\mathbf{F}: \mathbb{R}^n \to \mathbb{R}^m \to DF$

Vector fields $\vec{F}: \mathbb{R}^n \to \mathbb{R}^n$ $\to \text{curl } \vec{F}$, $\to \text{div } \vec{F}$

<u>Fundamental Thms</u> relate derivs & integrals

Gradient: $\vec{c}: \mathbb{R} \to \mathbb{R}^n$, $t \in [0,1]$, $f: \mathbb{R}^n \to \mathbb{R}$

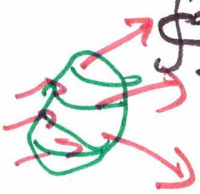
$$\underbrace{f(\vec{c}(1)) - f(\vec{c}(0))}_{\text{boundary } \vec{c}(0),\, \vec{c}(1)} = \int_{\vec{c}} \nabla f(\vec{c}) \cdot d\vec{c}$$

Curl: $\vec{c}$ boundary of $S \subset \mathbb{R}^3$ (or $\mathbb{R}^2$)

$$\oint \vec{F}(\vec{c}) \cdot d\vec{c} = \iint_S \text{curl } \vec{F} \cdot d\vec{S}$$

$\vec{n}\, du\, dv$

Divergence: S boundary of solid region $R \subset \mathbb{R}^3$



$$\oiint_S \vec{F} \cdot d\vec{S} = \iiint_R \text{div } \vec{F} \, dR$$

flux integral

Informal justifications:

value of fcn over a boundary = rate of fcn enclosed inside boundary.

Give some proofs of mult-variable theorems, using single-variable

Fund Thm of Calc:

$f: \mathbb{R} \to \mathbb{R}$ $\qquad \underline{f(b) - f(a)} = \int_{a}^{b} \underline{f'(t)}\, dt$

<u>Gradient Thm:</u> $f(\vec{c}(1)) - f(\vec{c}(0)) = \int_{\vec{c}} \nabla f \cdot d\vec{c}$

<u>Pf:</u> Chain Rule $\qquad f: \mathbb{R}^n \to \mathbb{R}$

$\vec{c}(t) = (x(t), y(t)) \qquad \vec{c}: \mathbb{R} \to \mathbb{R}^n$

$$\frac{d}{dt} f(\vec{c}(t)) = D(f \circ \vec{c})$$

$\mathbb{R} \to \mathbb{R}$

$$= Df_{\vec{c}(t)} \circ Dc_t$$

$$= \underset{1 \times n}{[Df_{c(t)}]} \cdot \underset{n \times 1}{[Dc_t]}$$

$$= [\vec{\nabla} f(\vec{c}(t))] \cdot [\vec{c}\,'(t)]$$

$$\begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix} \cdot \begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix}$$

$$= \frac{\partial f}{\partial x} x'(t) + \frac{\partial f}{\partial y} y'(t)$$

$f(\vec{v} + \vec{h}) \approx$

$\approx f(\vec{v}) + [Df_{\vec{v}}] \cdot [h]$

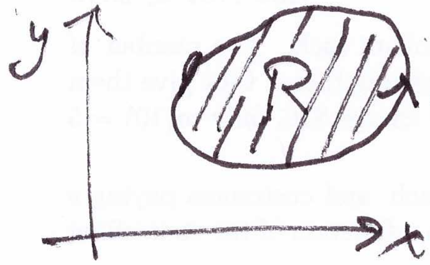
$[\cdot \cdot \cdot][:]$

$$\int_{\vec{c}} \vec{\nabla} f(\vec{c}(\;)) \cdot d\vec{c} = \int_{t=0}^{1} \underbrace{\vec{\nabla} f(\vec{c}(t)) \cdot \vec{c}\,'(t)}_{\text{Chain Rule}}\, dt$$

$$= \int_{t=0}^{1} \frac{d}{dt} \underbrace{f(\vec{c}(t))}\, dt$$

$$\overset{\text{Single-Var Fund Thm Calc}}{=} f(\vec{c}(1)) - f(\vec{c}(0))$$

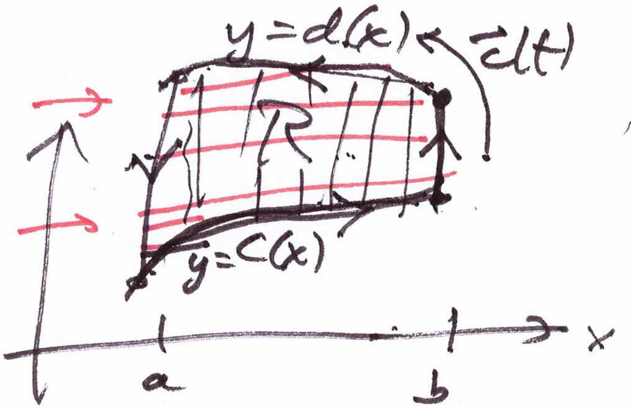
Prove Curl Thm for $\mathbb{R}^2$

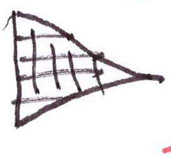


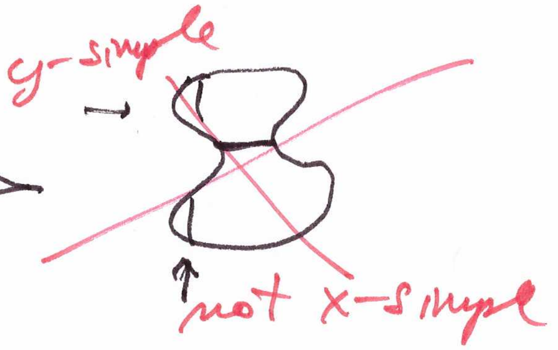
$$\oint \vec{F}(\vec{c}) \cdot d\vec{c} = \iint_R \underbrace{\text{curl}\,\vec{F}}_{\text{scalar}} \, \underbrace{dR}_{\text{area}}$$

scalar

Assume R is x-simple & y-simple







$$R = \begin{Bmatrix} a \le x \le b \\ c(x) \le y \le d(x) \end{Bmatrix}$$ boundary

$$\vec{F}(x,y) = (p(x,y), q(x,y))$$

boundary bottom $\quad \vec{c}_1(t) = (\overset{x(t)}{t}, \overset{y(t)}{c(t)}) \qquad a \le t \le b$

$$\vec{c}_2(t) = (\underset{x'=0}{b}, t) \qquad c(b) \le t \le d(b)$$

$$\vec{c}_3^{\,rev}(t) = (t, d(t)) \qquad a \le t \le b$$

$$c_4^{rev}(t) = (\underset{x'=0}{a}, t) \qquad c(a) \le t \le d(a)$$

$$\iint \text{curl}\, F \; dR = \iint_R \frac{\partial q}{\partial x} - \frac{\partial p}{\partial y} \, dx\, dy$$

$$= -\underbrace{\int_{x=a}^{b}\left[\int_{y=c(x)}^{d(x)} \frac{\partial p}{\partial y}(x,y)\, dy\right] dx}_{\text{FTC.}} + \underbrace{\iint_R \frac{\partial q}{\partial x}\, dx\, dy}_{\text{y-simple}}$$

$$= -\int_{x=a}^{b}\left[\, p(x,d(x)) - p(x,c(x))\right] dx + \int \text{''}\, q$$

$$= \int_{x=a}^{b} p(x,c(x)) - p(x,d(x))\, dx$$

$$\oint \vec{F}(\vec{c})\cdot d\vec{c} = \int_{c_1+c_2+c_3+c_4} (p(x(t),y(t)),\, q(x(t),y(t))) \cdot (x'(t), y'(t))\, dt$$

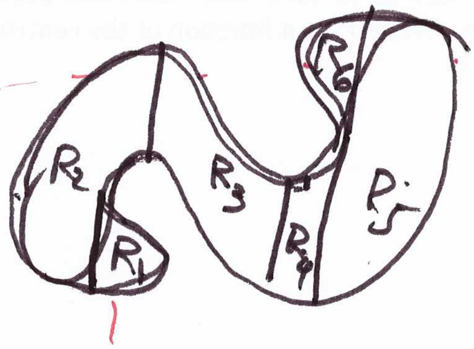
$$= \underbrace{\int_{\vec{c}} p(x,y)\, x'(t)\, dt} + \int_{\vec{c}} q(x,y)\, y'(t)\, dt$$

$$= \int_{t=a}^{b} p(t, c(t)) \cdot 1\, dt + \int q$$

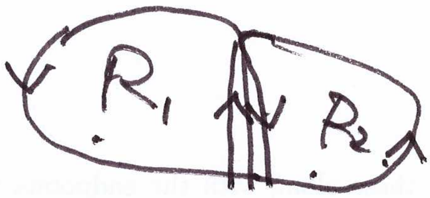
(c_1)

$$- \int_{t=a}^{b} p(t, d(t)) \cdot 1\, dt$$

This proof is valid for all R by splitting into simple regions.

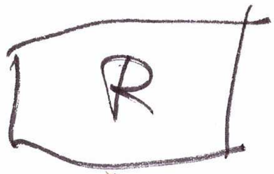


Thm is true for R_i's then true for R



Proof of Div Thm for $\mathbb{R}^2$

Assume R simple



$$\int_{\mathcal{C}} \vec{F} \cdot d\vec{n} = \iint_R \operatorname{div} \vec{F} \, dR$$

$$\iint_R \left(\frac{\partial p}{\partial x} + \frac{\partial q}{\partial y} \right) dx \, dy$$

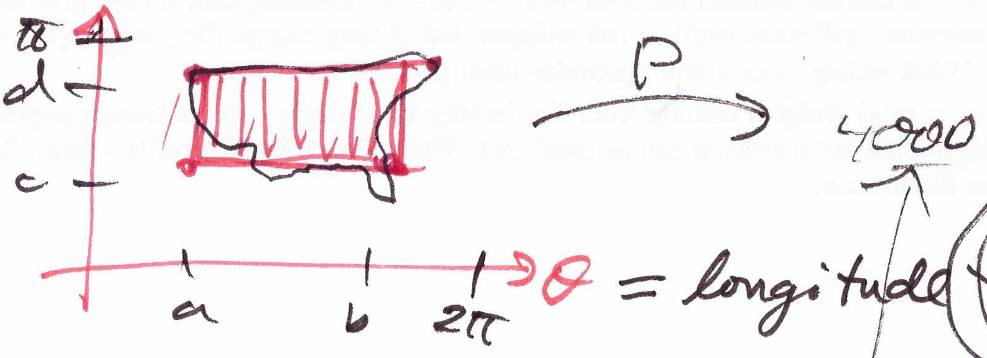
$$d\vec{n} = \vec{n}(t) \, dt$$

$$= (-y'(t), x'(t)) \, dt$$

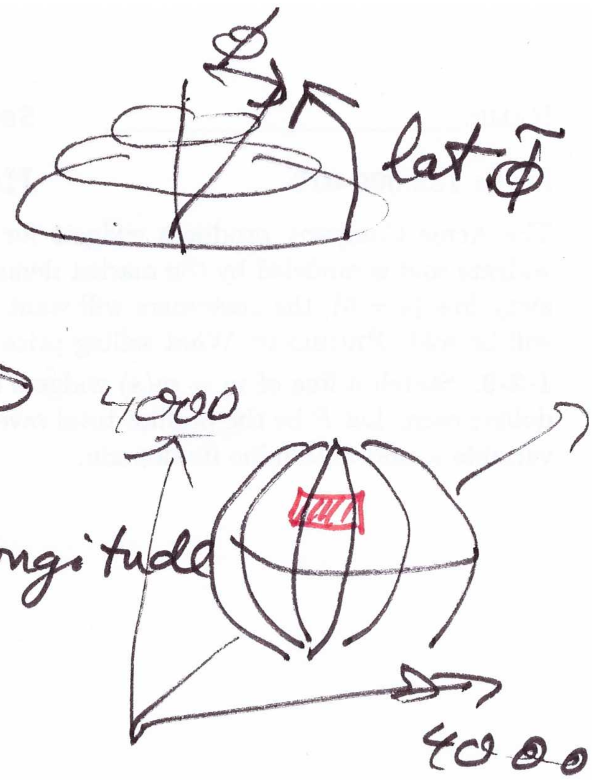
normal to curve $\vec{c}(t)$

WHW 7 #6(c)

$\varphi = 90° - \text{latitude}$



$\theta = \text{longitude}$



Spherical: $P(\theta, \varphi)$

$$\text{Area} = \int_{\theta=a}^{b}\int_{\varphi=c}^{d} 1 \left|\frac{\partial P}{\partial \theta} \times \frac{\partial P}{\partial \varphi}\right| d\theta\, d\varphi$$

stretching

$\rho^2 \sin\phi \, d\theta \, d\phi$

const

$\rho = 4000$ mi

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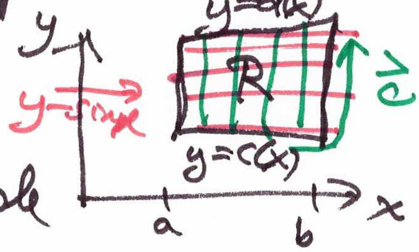
In $\mathbb{R}^2$: Curl Theorem, Divergence Thm

Proof using direct computation for simple region R



$$R=\left\{\begin{array}{l} a\le x\le b \\ c(x)\le y\le d(x)\end{array}\right\}$$ x-simple

also y-simple (any R can be cut up into simple R_i's)

Curl:

$$\underbrace{\oint_{\vec{c}} \vec{F}\cdot d\vec{c}}_{\text{circulation}} = \underbrace{\iint_R \operatorname{curl}\vec{F}\, dR}_{\text{int of rate of circ}}$$

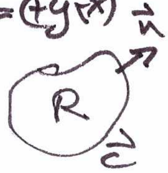
$$\vec{F}=(p,q) \qquad \vec{c}(t)=(x(t),y(t))$$

$$\oint_{t=0}^{1} \underbrace{p\,x'(t)}_{\text{use x-simple param of R}} + \underbrace{q\,y'(t)}_{\text{use y-simple}}\,dt = \iint_R \underbrace{\frac{\partial q}{\partial x}}_{\text{y-simple}} - \underbrace{\frac{\partial p}{\partial y}}_{\text{x-simple}}\,dx\,dy$$

Div:

$$\underbrace{\oint \vec{F}\cdot d\vec{n}}_{\text{net outflow of }\vec{F}\text{ from region across boundary }\vec{c}} = \underbrace{\iint_R \operatorname{div}\vec{F}\; dR}_{\text{int of rate of flux of }\vec{F}\text{ over }R}$$

$\vec{n}=(+y'(t)$



$$\int_{t=0}^{1} +p\,y'(t) - q\,x'(t)\,dt = \iint \frac{\partial p}{\partial x}+\frac{\partial q}{\partial y}\,dx\,dy$$

These are more than similar!

Equivalent! <u>Green's Theorem</u>

$$(!)\quad \int p\,x' + q\,y'\,dt = \iint \frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\,dt$$

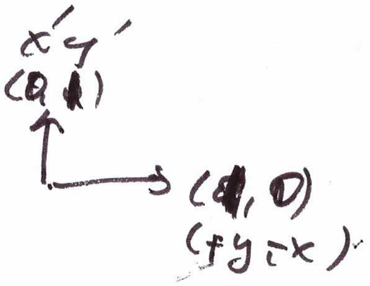
$$\int -\tilde{q}\,x' + \tilde{p}\,y'\,dt = \iint \frac{\partial \tilde{p}}{\partial x} + \frac{\partial \tilde{q}}{\partial y}\,dt$$

$-(-p) \quad +(+q) \qquad +q \quad -p$

$$\tilde{F} = (\tilde{p}, \tilde{q}) = (+q, -p)$$

$\vec{c}\,' = (x', y')$

$(y', -x') = \vec{n}$



rate of circ of $\vec{F}$ over surface S

In $\mathbb{R}^3$:

Curl Thm: $\oint_{\vec{c}} \vec{F} \cdot d\vec{c} = \iint_{S} \operatorname{curl} \vec{F} \cdot d\vec{S}$ (normal vector × area)

flux of curl $\vec{F}$ through S

Proof? (Using previous Fund Thm of Calc: Single-variable & Green's Thm)

Assume S is a graph

$S = \{(x, y, z(x,y)) \text{ for } (x,y) \in S^* \subset \mathbb{R}^2\}$

height function — parameter region

boundary $\vec{c}(t) = (x(t), y(t), z(x(t), y(t)))$

boundary $\vec{c}^*(t) = (x(t), y(t))$

LHS: $\vec{F} = (p, q, r)$

$\oint_c \vec{F}(\vec{c}) \cdot d\vec{c} = \int px' + qy' + rz' \, dt$

$\frac{d}{dt} z(x(t), y(t)) \underset{\text{Chain Rule}}{=} \frac{\partial z}{\partial x} x' + \frac{\partial z}{\partial y} y'$

$\mathbb{R} \xrightarrow{\vec{c}^*} \mathbb{R}^2 \xrightarrow{z} \mathbb{R}_{z(x,y)}$

$D(z \circ \vec{c}^*) = [Dz] \cdot [D\vec{c}] = \nabla z \cdot \vec{c}'$

$= \begin{bmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \end{bmatrix} \cdot \begin{bmatrix} x' \\ y' \end{bmatrix} = \frac{\partial z}{\partial x} x' + \frac{\partial z}{\partial y} y'$

LHS

$$\oint \vec{F}\cdot d\vec{c} = \int_t p x' + q y' + r \frac{\partial z}{\partial x} x' + r \frac{\partial z}{\partial y} y' \, dt$$

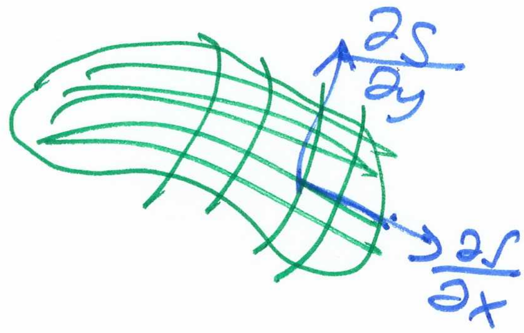
RHS

$$\iint_S \text{curl}\, \vec{F} \cdot d\vec{S} = \iint_{S^*} (\text{curl}\, \vec{F}) \cdot \underbrace{\left(\frac{\partial S}{\partial x} \times \frac{\partial S}{\partial y}\right)}\, dx\, dy$$

$$S(x,y) = (x, y, z(x,y))$$

$$\frac{\partial S}{\partial x} = \left(1, 0, \frac{\partial z}{\partial x}\right)$$

$$\frac{\partial S}{\partial y} = \left(0, 1, \frac{\partial z}{\partial y}\right)$$



$$\frac{\partial S}{\partial x} \times \frac{\partial S}{\partial y} = \left(-\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1\right)$$

$$\text{LHS} = \oint \cdots \xrightarrow[\text{Curl Thm for } \mathbb{R}^2]{\text{Green's Thm}} \iint = \text{RHS}$$

Like induction:

$$\text{Single-var FTC} \Longrightarrow \overset{\text{Green's Thm}}{\text{Curl Thm in } \mathbb{R}^2}$$

$$\text{Curl Thm in } \mathbb{R}^2 \Longrightarrow \overset{\text{Stokes Thm}}{\text{Curl Thm in } \mathbb{R}^3}$$

Div Thm (Gauss Thm)

$$\iint_S \underset{\text{vec}}{\vec{F}} \cdot \underset{\text{vec}}{d\vec{S}} = \iiint_R \underset{\text{scalar}}{\text{div}\, \vec{F}}\; \underset{dx\,dy\,dz\ \text{scalar}}{dR}$$