Lect 1

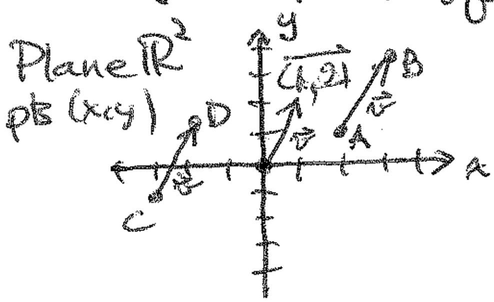
I. Vectors in $\mathbb{R}^2$

II. Orthogonal decomposition & $\vec{v}^{\perp}$

III. Determinant & signed area

0. Welcome. Course Info. Quiz 1: cosines, shadow

I. <u>Vector</u> $\vec{v}$ = arrow in the plane with length & direction
Same arrow at different locations is same vector
(Can think of $\vec{v}$ as shift motion of whole plane.)



Denote: $\vec{v} = \overrightarrow{AB} = \overrightarrow{CD}$

<u>Standard form</u>:

$$\vec{v} = \overrightarrow{(0,0)\,(v_1, v_2)} = \overrightarrow{(v_1, v_2)}$$

$\vec{v} = (1,2)$

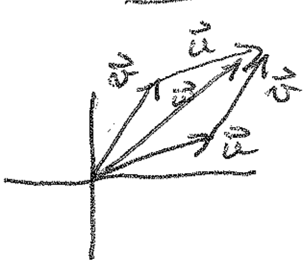
Draw with start at origin $(0,0)$
denote by <u>endpoint alone</u>

<u>Scalar multiple</u>: stretch or "scale" a vector
by a number s, a "scalar"

$$\vec{v} = \overrightarrow{(v_1, v_2)}, \quad s\vec{v} \overset{\text{def}}{=} (sv_1, sv_2)$$

$$\vec{v} = \overrightarrow{(1,2)}, \quad \tfrac{3}{2}\vec{v} = \overrightarrow{(\tfrac{3}{2}, 3)}, \quad -\vec{v} = -1\vec{v} = \overrightarrow{(-2,-3)}$$

<u>Add vectors</u> $\vec{u} + \vec{v} = \vec{w}$ compound motion
first shift by $\vec{u}$, then by $\vec{v}$



In standard form:

$$\vec{u} + \vec{v} = \overrightarrow{(u_1, u_2)} + \overrightarrow{(v_1, v_2)} = \overrightarrow{(u_1 + v_1,\ u_2 + v_2)}$$

$$\overrightarrow{(2,3)} + \overrightarrow{(1,2)} = \overrightarrow{(3,5)}$$

<u>Standard basis vectors</u>: $\vec{i} = \overrightarrow{(1,0)}$, $\vec{j} = \overrightarrow{(0,1)}$

$$\vec{v} = \overrightarrow{(v_1, v_2)} = v_1\overrightarrow{(1,0)} + v_2\overrightarrow{(0,1)} = v_1\vec{i} + v_2\vec{j}$$

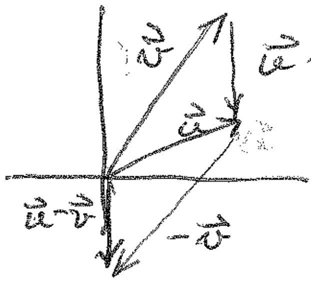
Note usual arithmetic properties

$s(\vec{u}+\vec{v}) = s\vec{u}+s\vec{v}$, $\quad (s+t)\vec{v} = s\vec{v}+t\vec{v}$

$\vec{u}+\vec{v} = \vec{v}+\vec{u}$ (Prove by looking at coordinates on both sides.)

Subtract: $\vec{u}-\vec{v} = \vec{u}+(-\vec{v})$

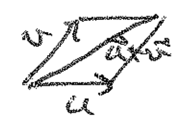
vector from end of $\vec{v}$ to end of $\vec{u}$



(Note: $\vec{v}+(\vec{u}-\vec{v}) = \vec{u}$

Summary: $s\vec{v}$ = stretch $\vec{v}$ by scale factor s

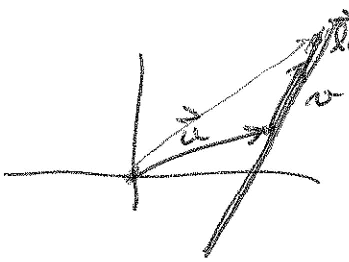
$-\vec{v}$ = opposite arrow

$\vec{u}+\vec{v}$ = diagonal of parallelogram 

$\vec{u}-\vec{v}$ = arrow from $\vec{v}$ to $\vec{u}$

Parametrized Line: $\vec{l}(t) = \vec{u}+t\vec{v}$

Position at time t, starting at $\vec{u}$, moving with velocity $\vec{v}$. Points of line = endpoint of $\vec{l}(t)$ in standard form

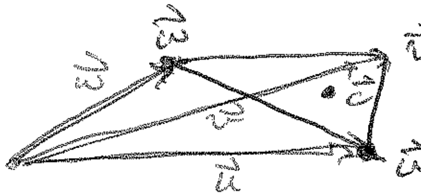


Midpoint between $\vec{u},\vec{v}$ $= \vec{v}+\frac{1}{2}(\vec{u}-\vec{v})$

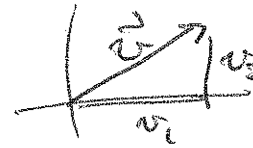
$= \vec{v}+\frac{1}{2}\vec{u}-\frac{1}{2}\vec{v} = \frac{1}{2}\vec{v}+\frac{1}{2}\vec{u}$ average.

Centroid of triangle

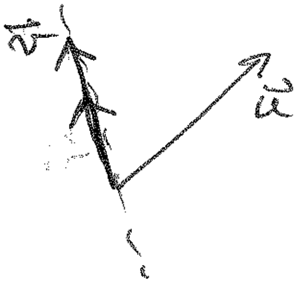
$\vec{c} = \frac{1}{3}(\vec{u}+\vec{v}+\vec{w})$



II. Length $|\vec{v}| = \sqrt{v_1^2 + v_2^2}$



Dot product: $\vec{u} \cdot \vec{v}$ = number (scalar) $\neq \cancel{|\vec{u}||\vec{v}|}$



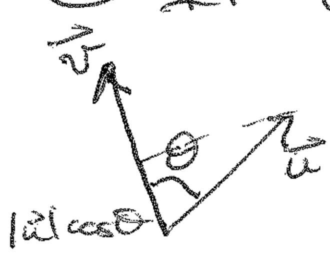
Let $\vec{p}$ = perpendicular projection of $\vec{u}$ in direction $\vec{v}$

$$\vec{u} \cdot \vec{v} \overset{def}{=} |\vec{p}|\,|\vec{v}|$$

i.e. project $\vec{u}$ in same direction as $\vec{v}$ then multiply the two lengths.

Then:

- If $\vec{u} = \vec{v}$ then $\vec{p} = \vec{v}$, $\vec{v} \cdot \vec{v} = |\vec{v}|^2$
- If $\vec{u} \perp \vec{v}$ perpendicular (orthogonal) then $\vec{p} = \vec{0}$ so $\vec{u} \cdot \vec{v} = 0 \cdot |\vec{v}| = 0$
- If $\theta_{\vec{u}\vec{v}}$ = angle between vectors

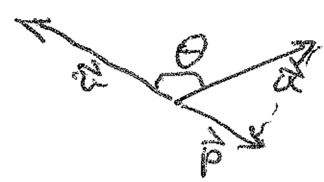


then $|\vec{p}| = |\vec{u}| \cos\theta_{\vec{u}\vec{v}}$

$$\vec{u} \cdot \vec{v} = |\vec{u}|\,|\vec{v}| \cos\theta_{\vec{u}\vec{v}}$$

- For obtuse angle $\theta_{\vec{u}\vec{v}} > 90°$

define $\vec{u} \cdot \vec{v} \overset{def}{=} -|\vec{p}|\,|\vec{v}|$



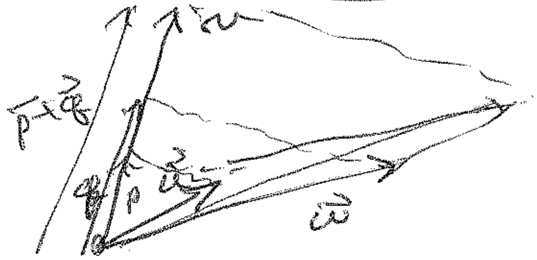
since $\vec{p}$ is opposite $\vec{v}$.

This keeps formula $\vec{u} \cdot \vec{v} = |\vec{u}|\,|\vec{v}| \cos\theta_{\vec{u}\vec{v}}$, since $\cos\theta < 0$ for obtuse θ

Commutative law: $\vec{u} \cdot \vec{v} = |\vec{u}|\,|\vec{v}| \cos\theta_{\vec{u}\vec{v}}$

$= |\vec{v}|\,|\vec{u}| \cos\theta_{\vec{v}\vec{u}} = \vec{v} \cdot \vec{u}$

Distributive Law: $(\vec{u}+\vec{w})\cdot\vec{v} = \vec{u}\cdot\vec{v}+\vec{w}\cdot\vec{v}$ 4



perp proj ~~of~~ of $\vec{u}$ is $\vec{p}$

of $\vec{w}$ is $\vec{q}$

of $\vec{u}+\vec{w}$ is $\vec{p}+\vec{q}$

all parallel along $\vec{v}$

$|\vec{p}+\vec{q}| = |\vec{p}|+|\vec{q}|$

Coordinate formula:

$$\vec{u}\cdot\vec{v} = (u_1\vec{i}+u_2\vec{j})\cdot(v_1\vec{i}+v_2\vec{j})$$

$$= u_1u_2\,\vec{i}\cdot\vec{i} + u_1v_2\,\vec{i}\cdot\vec{j} + u_2v_1\,\vec{j}\cdot\vec{i} + v_1v_2\,\vec{j}\cdot\vec{j}$$

$$= u_1u_2 + v_1v_2$$

Example: Find length & angle for $\vec{u}=\langle 2,3\rangle$, $\vec{v}=\langle 1,2\rangle$

$$|\vec{u}| = \sqrt{\vec{u}\cdot\vec{u}} = \sqrt{2^2+3^2} = \sqrt{13}$$

$$|\vec{v}| = \sqrt{\vec{v}\cdot\vec{v}} = \sqrt{2^2+1^2} = \sqrt{5}$$

$$\vec{u}\cdot\vec{v} = 2\cdot 2+3\cdot 1 = 7 = |\vec{u}|\,|\vec{v}|\cos\theta_{\vec{u}\vec{v}}$$

$$\cos\theta_{\vec{u}\vec{v}} = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}|\,|\vec{v}|} = \frac{7}{\sqrt{13}\sqrt{5}}$$

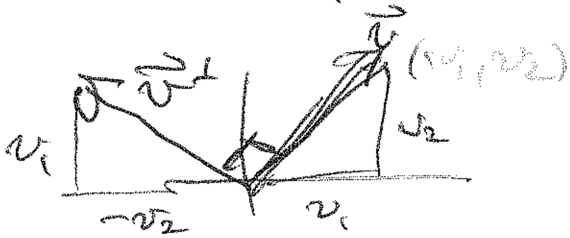
$$\theta_{\vec{u}\vec{v}} = \arccos\frac{7}{\sqrt{13}\sqrt{5}} \approx 0.519 \text{ rad}$$

$$29.7^\circ$$

III. Orthogonal of a vector:

$\vec{v} = (v_1, v_2)$

$\vec{v}^\perp = (-v_2, v_1)$



Proof: $\vec{v}^\perp \cdot \vec{v} = -v_2 v_1 + v_1 v_2 = 0$

Dot prod $= 0 \iff \cos\theta = 0$

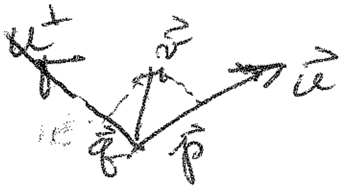
vectors orthogonal

Orthogonal decomposition:

If $|\vec{u}| = 1$, then $\vec{v} \cdot \vec{u} = |\vec{p}|\,|\vec{u}| = |\vec{p}|$

unit vector

$\vec{p}$ = project $\vec{v}$ to $\vec{u}$

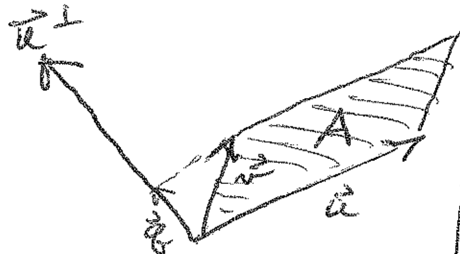


Also $\vec{v} \cdot \vec{u}^\perp = |\vec{q}|$ project $\vec{v}$ to $\vec{u}^\perp$

If $|\vec{u}| \neq 1$, replace $\vec{u}$ by $\dfrac{\vec{u}}{|\vec{u}|}$ unit vector.

Area of parallelogram:

spanned by $\vec{u}, \vec{v}$



Area = base × height

base = $|\vec{u}|$

height = $|\vec{q}|$ project $\vec{v}$ to $\vec{u}^\perp$

$= \left|\vec{v} \cdot \dfrac{\vec{u}^\perp}{|\vec{u}^\perp|}\right|$ ← unit vector $= \dfrac{|\vec{v} \cdot \vec{u}^\perp|}{|\vec{u}|}$

area $= |\vec{u}| \cdot \dfrac{|\vec{v} \cdot \vec{u}^\perp|}{|\vec{u}|} = |\vec{v} \cdot \vec{u}^\perp|$

Note: $\det(\vec{u}, \vec{v}) > 0$ when $\theta_{\vec{u}^\perp, \vec{v}} < 90°$

$\vec{v}$ on counterclockwise

Define signed area $\det\begin{pmatrix}\vec{u}\\ \vec{v}\end{pmatrix} = \vec{v} \cdot \vec{u}^\perp = \vec{u}^\perp \cdot \vec{v}$

$= (-u_2, u_1) \cdot (v_1, v_2) = -u_2 v_1 + u_1 v_2 = u_1 v_2 - u_2 v_1$