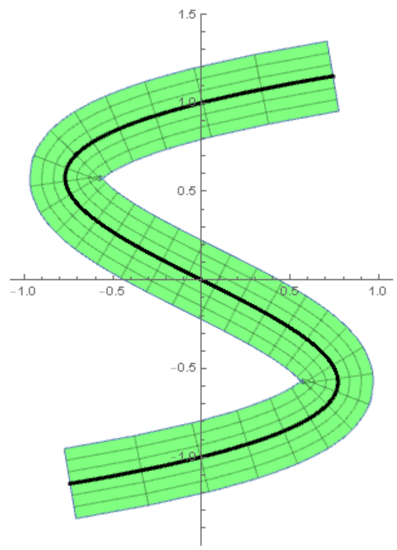
**Math 254H** # Extra Credit Homework **Due 10/31/2019**

Do any of these problems for up to 20 extra points to raise your course grade.

For a curve $\vec{c}(t) = (x(t), y(t))$ with $a \le t \le b$, define its thickened curve $C$ as the union of all line segments of a fixed length cutting perpendicularly across it:




Here the graph $x = 2(y^3 - y)$ is parametrized as $\vec{c}(t) = (2(t^3 - t), t)$ with $-1.15 \le t \le 1.15$, and thickened to a width of 0.4 to resemble a letter S.

**1.** A thickened curve is a region in $\mathbb{R}^2$. Parametrize it by a function $C : \mathbb{R}^2 \to \mathbb{R}^2$ of the form $C(t, s) = (x(t, s), y(t, s))$.

Here the original curve is $\vec{c}(t) = C(t, 0)$ for $a \le t \le b$, and the perpendicular segments are $C(t_\circ, s)$ for fixed $t_\circ$ and $-w \le s \le w$, giving width $2w$. Your formulas for $x(t, s), y(t, s)$ should be in terms of the original $x(t), y(t)$, and variables $t, s$.

*Hint:* Use the counterclockwise normal vector $\vec{c}\,'(t)^\perp = (-y'(t), x(t))$, scaling its length.

**2.** Choose appropriate curves to represent letters, and plot them to spell a word.

The best software for this is Mathematica, which you can install for free from the MSU Computer Store: `https://techstore.msu.edu/mathematica-installation-and-use-information` . See the example commands in the accompanying `.nb` file.

There seems no direct way to get WolframAlpha to plot parametric regions, but a work-around is: `3d parametric plot {x(t,s),y(t,s),0} for t = a to b, s = -w to w)`, which will show the region in the $xy$-plane of $\mathbb{R}^3$. If you pay a couple of dollars for W|A Pro, you can rotate to view from above.

You may also use other web apps that can show these regions.

**3.** Draw the above letters in 3D: raise them in the $z$-direction, and add vertical side surfaces around them.

**4.** Proposition: The area $A$ of a thickened curve $C$ equals the length $\ell$ of the curve, times width $2w$.

Give as precise an argument as you can for this. First prove it by elementary geometry for a circle or other simple curves. Next, give an approximation argument, cutting the region into small trapezoids, and adding their areas.

Finally, give an analytic proof using the Substitution Rule to compute $A = \iint_C 1\, dx\, dy$. *Hints:* To simplify the calculations, assume $\vec{c}(t)$ travels along the curve at constant speed $|\vec{c}\,'(t)| = 1$, and prove $\vec{c}\,' \cdot \vec{c}\,'' = 0$.