Weekly Homework 8 – Hint

Here is a model for the proof of the Conservative Vector Field Theorem.

State the full theorem before the proof.

Proof: (i) $\Rightarrow$ (ii): Suppose $\vec{F}$ is conservative, meaning $\vec{F} = \nabla f$ for some potential function $f(x, y)$. Let $\mathbf{c}_1, \mathbf{c}_2$ be any two paths between the same endpoints, so that $\mathbf{c}_1(0) = \mathbf{c}_2(0)$ and $\mathbf{c}_1(1) = \mathbf{c}_2(1)$. Then we have:

$$\begin{aligned} \textstyle\int \vec{F}(\mathbf{c}_1) \cdot d\mathbf{c}_1 &= f(\mathbf{c}_1(1)) - f(\mathbf{c}_1(0)) && \text{by the Gradient Theorem} \\ &= f(\mathbf{c}_2(1)) - f(\mathbf{c}_2(0)) && \text{since } \mathbf{c}_1 \text{ and } \mathbf{c}_2 \text{ have same endpoints} \\ &= \textstyle\int \vec{F}(\mathbf{c}_2) \cdot d\mathbf{c}_2 && \text{by the Gradient Theorem.} \end{aligned}$$

Therefore $\int \vec{F}(\mathbf{c}_1) \cdot d\mathbf{c}_1 = \int \vec{F}(\mathbf{c}_2) \cdot d\mathbf{c}_2$; by definition $\vec{F}$ is path-independent.

Style notes: Assume the hypothesis (setup) at the beginning. The conclusion is a formula, so start with the left side and transform it step by step into the right side, justifying each step by a theorem or assumption.

In your rough draft, you may work from both sides of the formula, but in your final proof, arrange all the equalities to go from the left side to the right side.