Math 254H **Weekly Homework 6** **Due 10/7/2019**

Background: The Mean Value Theorem. This is an essential tool to prove basic facts about derivatives and integrals, making sure there can be no exceptions.

MEAN VALUE THEOREM 1: If a function $f:\mathbb{R}\to\mathbb{R}$ is differentiable on an interval $[a,b]$, then one of its tangent slopes inside the interval is equal to the secant slope across the interval: there is some $c\in(a,b)$ with

$$f'(c) \;=\; \frac{f(b)-f(a)}{b-a}\,.$$

Proof. Consider the vertical difference between the secant line and the graph $y=f(x)$:

$$g(x) \;=\; \tfrac{f(b)-f(a)}{b-a}(x-a)+f(a)-f(x).$$

This function is differentiable over the interval with end values $g(a)=g(b)=0$, so it has a max or min point $c\in(a,b)$, and we must have $g'(c)=0$. (This is Rolle's Theorem, easy to show from the definition $g'(c)=\lim_{x\to 0}\frac{g(x)-g(c)}{x-c}$.) But $g'(x)=\frac{f(b)-f(a)}{b-a}-f'(x)$, so:

$$0 \;=\; g'(c) \;=\; \tfrac{f(b)-f(a)}{b-a}-f'(c).$$

MEAN VALUE THEOREM 2: If a function $f:\mathbb{R}\to\mathbb{R}$ is continuous on an interval $[a,b]$, then one of its values inside the interval is equal to the average value over the interval: there is some $c\in(a,b)$ with

$$f(c) \;=\; \frac{1}{b-a}\int_a^b f(t)\,dt\,.$$

Proof. Consider the integral function $F(x)=\int_a^x f(t)\,dt$. The First Fundamental Theorem of Calculus says this has derivative $F'(x)=f(x)$ for $x\in[a,b]$. (This is because the rate of change of the area under the graph above $[a,x]$ is equal to the height at the right edge.)

Now, applying MVT 1 to $F(x)$, there is some $c\in(a,b)$ with:

$$f(c) \;=\; F'(c) \;=\; \tfrac{F(b)-F(a)}{b-a} \;=\; \tfrac{1}{b-a}\int_a^b f(t)\,dt.$$

To illustrate the use of the Mean Value Theorem, we need it to prove:

MONOTONICTY: If $f'(x)\ge 0$ for all $x\in[a,b]$, then f is increasing on $[a,b]$.
Proof. Applying MVT 1 to any subinterval $[\tilde a,\tilde b]\subset[a,b]$ gives $c\in(\tilde a,\tilde b)$ with:

$$\tfrac{f(\tilde b)-f(\tilde a)}{\tilde b-\tilde a} \;=\; f'(c) \;\ge\; 0.$$

Multiplying both sides of the inequality by $\tilde b-\tilde a>0$ shows $f(\tilde b)-f(\tilde a)\ge 0$ and $f(\tilde b)\ge f(\tilde a)$ for any $a\le\tilde a<\tilde b\le b$.

Problems

0. First, prove the familiar UNIQUENESS THEOREM: If f_1, f_2 have the same derivative $f_1'(x) = f_2'(x)$ for all $x \in [a,b]$, then $f_1(x) = f_2(x) + C$ for a constant C. *Hint:* Apply MVT 1 to the function $g(x) = f_1(x) - f_2(x)$ over any subinterval $[\tilde{a}, \tilde{b}] \subset [a,b]$.

Next, recall that for a vector field $\vec{F}(x,y) = (p(x,y), q(x,y))$, we defined curl $\vec{F}(x,y)$ as a kind of derivative which measures the counterclockwise rotation of $\vec{F}$ near each point (x,y). We defined this in two ways: geometrically, it is the rate of circulation around a small loop near (x,y), per unit area enclosed; computationally, we add a counterclockwise contribution from the change in the vertical component of $\vec{F}$ over a short horizontal increment, $\frac{\partial q}{\partial x}$, and a clockwise contribution from the change in the horizontal component of $\vec{F}$ over a short vertical increment, $\frac{\partial p}{\partial y}$. The main goal is to prove the equivalence of these two definitions.

THEOREM: For given $x, y, \Delta x, \Delta y$, let $\mathbf{c}(t)$ be a closed rectangular curve from (x,y) to $(x{+}\Delta x, y)$ to $(x{+}\Delta x, y{+}\Delta y)$ to $(x, y{+}\Delta y)$ back to (x,y). Then:

$$\lim_{\Delta x, \Delta y \to 0} \frac{1}{\Delta x \Delta y} \oint_{\mathbf{c}} \vec{F}(\mathbf{c}) \cdot d\mathbf{c} \quad = \quad \frac{\partial q}{\partial x}(x,y) - \frac{\partial p}{\partial y}(x,y).$$

1. Parametrize $\mathbf{c}$ in four linear segments $\mathbf{c}_1, \ldots, \mathbf{c}_4$, for example $\mathbf{c}_1(t) = (x{+}t, y)$ for $t \in [0, \Delta x]$. Find their (constant) tangent vectors $\mathbf{c}_1'(t), \ldots, \mathbf{c}_4'(t)$. Split the line integral over the segments as $\oint = \int_1 + \cdots + \int_4$, and simplify each term as far as possible using the (x,y)-components of $\vec{F}$ and $\mathbf{c}$. (Remember that $x, y, \Delta x, \Delta y$ are constants throughout this computation.)

2. To make Step 4 work below, you should substitute the variable $\tilde{t} = \Delta x - t$ in $\int_3$ to make it match better with $\int_1$, and similarly for $\int_4$ matching with $\int_2$. Combine $\int_1 + \int_3$ into a single integral $\int_0^{\Delta x}$, and similarly $\int_2 + \int_4 = \int_0^{\Delta y}$.

3. Apply MVT 2 to the two integrals $\int_0^{\Delta x}$ and $\int_0^{\Delta y}$, obtaining difference quotients involving p and q. (No more integrals.)

4. Apply MVT 1 to each difference quotient, obtaining derivatives

$$-\frac{\partial p}{\partial y}(x + d_1, y + e_1) \quad \text{and} \quad \frac{\partial q}{\partial x}(x + d_2, y + e_2),$$

where $d_1, d_2 \in [x, x{+}\Delta x]$ and $e_1, e_2 \in [y, y{+}\Delta y]$. Now take the limit as $\Delta x, \Delta y \to 0$. To get the desired quantity, what assumption do you need about the functions $\frac{\partial p}{\partial y}(x,y), \frac{\partial q}{\partial x}(x,y)$?