Math 254H **Weekly Homework 5** **Due Mon Sep 30, 2019**

For a mapping $F(u,v) = (x(u,v), y(u,v))$, its derivative $L = DF_{(u_\circ,v_\circ)}$ at a point $(u_\circ, v_\circ)$ in the uv-plane is the linear function which gives the best affine approximation for (u,v) close to $(u_\circ, v_\circ)$, so that :

$$F(u,v) \;\approx\; F(u_\circ,v_\circ) + DF_{(u_\circ,v_\circ)}(u{-}u_\circ, v{-}v_\circ).$$

The linear mapping $DF_{(u_\circ,v_\circ)}$ is computed using its matrix, the *Jacobian matrix*:

$$[DF_{(u_\circ,v_\circ)}] \;=\; \begin{bmatrix} \nabla x(u_\circ,v_\circ) \\ \nabla y(u_\circ,v_\circ) \end{bmatrix} \;=\; \begin{bmatrix} \frac{\partial x}{\partial u}(u_\circ,v_\circ) & \frac{\partial x}{\partial u}(u_\circ,v_\circ) \\ \frac{\partial y}{\partial u}(u_\circ,v_\circ) & \frac{\partial y}{\partial u}(u_\circ,v_\circ) \end{bmatrix}.$$

This is a matrix of *constants* involving the constant values $u_\circ, v_\circ$, not variables u, v.

1. Consider the polar coordinate mapping $P : [0,\infty) \times [0,2\pi) \to \mathbb{R}^2$ given by:

$$P(r,\theta) \;=\; (r\cos(\theta), r\sin(\theta)).$$

Find the Jacobian matrix $[DP_{(r_\circ,\theta_\circ)}]$ at $(r_\circ,\theta_\circ) = (2, \frac{\pi}{6})$. Write the corresponding affine approximation $P(r,\theta) \approx A(r,\theta)$ where:

$$A(r,\theta) = P(r_\circ,\theta_\circ) + DP_{(r_\circ,\theta_\circ)}(r{-}r_\circ, \theta{-}\theta_\circ).$$

Multiply out the matrices to get its x and y-components explicitly.

2. Write the above Jacobian matrix as a composition of a rotation and a dilation (given by a diagonal matrix $\left[\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right]$).

3. Draw a polar $r\theta$-grid on the xy-plane in colored pen, the curves $P(1,\theta), P(2,\theta), \ldots$ and $P(r,0), P(r,\frac{\pi}{6}), P(r,\frac{\pi}{3}), \ldots$, including the curves through the base point $P(2,\frac{\pi}{6})$.

Then pencil in the linear grid corresponding to the affine approximation: the lines $A(1,\theta), A(2,\theta), \ldots$ and $A(r,0), A(r,\frac{\pi}{6}), A(r,\frac{\pi}{3}), \ldots$. The point is, the affine grid approximates the polar grid near the base point.

Next, repeat the above exercises for the inversion mapping, which takes each vector (u,v) to a vector in the same direction, but with radius equal to the reciprocal of the original radius:

$$F(u,v) \;=\; \frac{(u,v)}{u^2+v^2}.$$

4. Compute the Jacobian and the affine approximation of $F(u,v)$ near the base point $(u_\circ,v_\circ) = (1,1)$.

5. Write $DF_{(1,1)}$ in terms of standard linear mappings. *Hint:* No rotation involved.

6. Get Wolfram Alpha to draw some of the uv-grid curves $F(1,v), F(2,v), \ldots$ and $F(u,1), F(u,2), \ldots$. Use algebra to show these are circles with centers on the axes. Draw a few curves in colored pen, including the ones through the base point $F(1,1)$. Then draw the grid of the linear approximation.