Math 254H **Weekly Homework 2** **Due Mon Sep 9, 2019**

A *linear mapping* $L : \mathbb{R}^n \to \mathbb{R}^m$ is a mapping compatible with vector addition and scalar multiplication, meaning for any vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ and scalars $s, t \in \mathbb{R}$, we have:

$$L(s\mathbf{u} + t\mathbf{v}) \;=\; s\,L(\mathbf{u}) + t\,L(\mathbf{v}).$$

For $L : \mathbb{R}^2 \to \mathbb{R}^2$, this means L is specified by the two outputs $L(\mathbf{i}) = L(1,0) = (a,b)$ and $L(\mathbf{j}) = L(0,1) = (c,d)$. For a general vector $(x,y) = x\,(1,0) + y\,(0,1)$, we have:

$$L(x,y) \;=\; x\,(a,b) + y\,(c,d) \;=\; (ax{+}cy,\; bx{+}dy),$$

so that a, b, c, d are slope constants, which we write in a 2×2 matrix:

$$[\,L\,] \;=\; \left[\begin{array}{c|c} a & c \\ b & d \end{array}\right].$$

When computing with matrices, we usually write a vector $\mathbf{v} = (x,y)$ in column form as $[\mathbf{v}] = \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$. We define matrix multiplication of the matrix $[L]$ times $[\mathbf{v}]$ so that it produces the output $[L] \cdot [\mathbf{v}] = [L(\mathbf{v})]$; that is:

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} \overset{\text{def}}{=} \begin{bmatrix} ax + cy \\ bx + dy \end{bmatrix} \;=\; \begin{bmatrix} (a,c) \cdot (x,y) \\ (b,d) \cdot (x,y) \end{bmatrix}.$$

That is, matrix multiplication takes dot product of the row vectors of $[L]$ with $\mathbf{v}$.

1. For two linear mappings $L_1, L_2 : \mathbb{R}^2 \to \mathbb{R}^2$, the composite function $L_3 = L_1 \circ L_2$ is defined by $L_3(\mathbf{v}) = L_1(L_2(\mathbf{v}))$. Prove that L_3 is also a linear mapping.

2. Given the matrices:

$$[\,L_1\,] \;=\; \left[\begin{array}{c|c} a_1 & c_1 \\ b_1 & d_1 \end{array}\right], \qquad [\,L_2\,] \;=\; \left[\begin{array}{c|c} a_2 & c_2 \\ b_2 & d_2 \end{array}\right],$$

find the matrix $[L_3]$ of the composition $L_3 = L_1 \circ L_2$. Interpret each entry of $[L_3]$ as a dot product of row or column vectors of $[L_1]$ and $[L_2]$.

3. Write the matrix of a general plane rotation Rot_θ, counterclockwise by angle θ.

4. Find the matrix of the composite mapping $\mathrm{Rot}_\alpha \circ \mathrm{Rot}_\beta$, and interpret it as a known linear mapping. *Hint:* Use the trigonometric angle addition identities.

5. Let $\mathbf{u}_\theta = (\cos(\theta), \sin(\theta))$ be the unit vector with angle θ from the x-axis. Let $\mathrm{Ref}_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be the orthogonal reflection of the plane which flips $\mathbf{u}_\theta$ and fixes the line perpendicular to $\mathbf{u}_\theta$. Find the matrix of Ref_θ. *Hint:* Recall that $\mathrm{Ref}_\theta(\mathbf{v}) = \mathbf{v} - 2\mathbf{p}$, where $\mathbf{p}$ is the orthogonal projection of $\mathbf{v}$ onto the direction of $\mathbf{u}_\theta$.

6. Find the matrix of the composite mapping $\mathrm{Ref}_\alpha \circ \mathrm{Ref}_\beta$, and interpret it as a known linear mapping.