1. If $L:\mathbb{R}^n\to\mathbb{R}^m$ is a linear function, then we have the exact formula:

$$L(\mathbf{a}+\mathbf{h}) \;=\; L(\mathbf{a})+L(\mathbf{h}),$$

so L gives its own affine approximation near any point $\mathbf{a}\in\mathbb{R}^n$, and $DL_{\mathbf{a}}=L$.

2a. The Jacobian derivatives of $f(x,y)$ and $P(r,\theta)=(r\cos(\theta),r\sin(\theta))$ are:

$$[Df_{(x,y)}]=\nabla f(x,y)=\left[\tfrac{\partial f}{\partial x},\tfrac{\partial f}{\partial y}\right],\qquad [DP_{(r,\theta)}]=\begin{bmatrix}\cos(\theta) & -r\sin(\theta)\\ \sin(\theta) & r\cos(\theta)\end{bmatrix}.$$

By the Chain Rule, the derivative of $g(r,\theta)=f(P(r,\theta))$ is the product matrix:

$$\begin{aligned}[Dg_{(r,\theta)}] &= \nabla g(r,\theta)=\left[\tfrac{\partial f}{\partial x}(P(r,\theta)),\tfrac{\partial f}{\partial y}(P(r,\theta))\right]\cdot\begin{bmatrix}\cos(\theta) & -r\sin(\theta)\\ \sin(\theta) & r\cos(\theta)\end{bmatrix}\\ &= \left[\tfrac{\partial f}{\partial x}\cos(\theta)+\tfrac{\partial f}{\partial y}\sin(\theta)\ ,-\tfrac{\partial f}{\partial x}r\sin(\theta)+\tfrac{\partial f}{\partial y}r\cos(\theta)\right]\end{aligned},$$

where $\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}$ are taken at $P(r,\theta)$.

2b. We get the same result in Leibnitz notation, writing $z=f(x,y)$, $x=r\cos(\theta)$, $y=r\sin(\theta)$ and computing:

$$\frac{\partial z}{\partial r} \;=\; \frac{\partial z}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial r} \;=\; \frac{\partial f}{\partial x}(x,y)\,\cos(\theta)+\frac{\partial f}{\partial y}(x,y)\,\sin(\theta),$$

and similarly for $\frac{\partial z}{\partial \theta}$.

3. For $\mathbf{c}(\theta)=P(r(\theta),\theta)$, we have:

$$\begin{aligned}\mathbf{c}'(\theta) &= [DP_{(r(\theta),\theta)}]\cdot[D(r(\theta),\theta))_\theta]\\ &= \begin{bmatrix}\cos(\theta) & -r(\theta)\sin(\theta)\\ \sin(\theta) & r(\theta)\cos(\theta)\end{bmatrix}\cdot\begin{bmatrix}r'(\theta)\\ 1\end{bmatrix}\\ &= \begin{bmatrix}\cos(\theta)r'(\theta)-r(\theta)\sin(\theta)\\ \sin(\theta)r'(\theta)+r(\theta)\cos(\theta)\end{bmatrix}\\ &= r'(\theta)\begin{bmatrix}\cos(\theta)\\ \sin(\theta)\end{bmatrix}+r(\theta)\begin{bmatrix}\cos(\theta)\\ \sin(\theta)\end{bmatrix}'\end{aligned}$$

The last expression agrees with the Product Rule applied to $\mathbf{c}(\theta)=r(\theta)(\cos(\theta),\sin(\theta))$.

4. For $m(x,y)=xy$ and $F(x,y)=(f(x,y),g(x,y))$, the Jacobians are:

$$[Dm_{x,y}]=\nabla m(x,y)=[\,y\ \ x\,],\qquad [DF_{(x,y)}]=\begin{bmatrix}\frac{\partial f}{\partial x} & \frac{\partial f}{\partial y}\\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y}\end{bmatrix}.$$

Letting $F(a,b) = (f(a,b), g(a,b)) = (f,g)$, the Chain Rule says:

$$\begin{aligned}
\nabla(fg)(a,b) &= [D(m\circ F)_{(a,b)}] \;=\; [Dm_{F(a,b)}]\cdot[DF_{(a,b)}] \\
&= \nabla m(f,g)\cdot \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix} \;=\; [\,g \quad f\,]\cdot \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix} \\
&= \left[g\,\tfrac{\partial f}{\partial x} + f\,\tfrac{\partial g}{\partial x}\ ,\ g\,\tfrac{\partial f}{\partial y} + f\,\tfrac{\partial g}{\partial y} \right] \\
&= g\,\left[\tfrac{\partial f}{\partial x}, \tfrac{\partial f}{\partial y} \right] + f\,\left[\tfrac{\partial g}{\partial x}, \tfrac{\partial g}{\partial y} \right] \\
&= g(a,b)\,\nabla f(a,b) + f(a,b)\,\nabla g(a,b).
\end{aligned}$$

5. Given $h(x,y) = f(x, g(x,y))$, we define $G(x,y) = (x, g(x,y))$ and compute:

$$\begin{aligned}
\nabla h(a,b) &= [D(f\circ G)_{(a,b)}] \;=\; [Df_{G(a,b)}]\cdot[DG_{(a,b)}] \\
&= \nabla f(a, g(a,b))\cdot \begin{bmatrix} 1 & 0 \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix} \\
&= \left[\tfrac{\partial f}{\partial x}(a,g(a,b)) + \tfrac{\partial f}{\partial y}(a,g(a,b))\,\tfrac{\partial g}{\partial x}(a,b)\ ,\ \tfrac{\partial f}{\partial y}(a,g(a,b))\,\tfrac{\partial g}{\partial y}(a,b) \right]
\end{aligned}$$

Notice that $\frac{\partial f}{\partial y}$ means the partial derivative of $f(x,y)$ with respect to the second variable, *before* substituting $g(x,y)$ for that variable.

This might be clearer in Leibnitz letter-notation:

$$z = f(x,y), \quad x = u, \quad y = g(u,v),$$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} = \frac{\partial f}{\partial x}(u, g(u,v)) + \frac{\partial f}{\partial y}(u, g(u,v))\,\frac{\partial g}{\partial u}(u,v),$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} = 0 + \frac{\partial f}{\partial y}(u, g(u,v))\,\frac{\partial g}{\partial v}(u,v),$$