Math 254H **Homework 4 Solutions** **Fall 2019**

1a. The projection mapping $P_{\mathbf{u}}(\mathbf{v}) = \frac{\mathbf{v}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\,\mathbf{u}$ has matrix:

$$[\mathrm{P}_{\mathbf{u}}] \;=\; [\; P_{\mathbf{u}}(\mathbf{i}) \mid P_{\mathbf{u}}(\mathbf{j}) \;] \;=\; \tfrac{1}{u_1^2+u_2^2}\begin{bmatrix} u_1^2 & u_1u_2 \\ u_1u_2 & u_2^2 \end{bmatrix} \;=\; \begin{bmatrix} u_1^2 & u_1u_2 \\ u_1u_2 & u_2^2 \end{bmatrix},$$

where the scalar next to the matrix multiplies every entry, and $|\mathbf{u}| = u_1^2 + u_2^2 = 1$.

1b. In geometric terms, we can compute:

$$P_{\mathbf{u}}(P_{\mathbf{u}}(\mathbf{v}) \;=\; \frac{\left(\frac{\mathbf{v}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\,\mathbf{u}\right)\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\,\mathbf{u} \;=\; \frac{\mathbf{v}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\frac{\mathbf{u}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\,\mathbf{u} \;=\; P_{\mathbf{u}}(\mathbf{v}).$$

Thus $\mathrm{Proj}_{\mathbf{u}} \circ \mathrm{Proj}_{\mathbf{u}} = \mathrm{Proj}_{\mathbf{u}}$, and it is geometrically clear that the second projection to direction $\mathbf{u}$ has no effect on the already projected vectors. The product matrix is:

$$[P_{\mathbf{u}}]\cdot[P_{\mathbf{u}}] \;=\; \begin{bmatrix} u_1^2 & u_1u_2 \\ u_1u_2 & u_2^2 \end{bmatrix}\cdot\begin{bmatrix} u_1^2 & u_1u_2 \\ u_1u_2 & u_2^2 \end{bmatrix} \;=\; \begin{bmatrix} u_1^4{+}u_1^2u_2^2 & -u_1u_2^3 - u_1^3u_2 \\ -u_1u_2^3 - u_1^3u_2 & u_1^2u_2^2 + u_1^4 \end{bmatrix}$$

The identity $u_1^2 + u_2^2 = 1$ easily reduces this to the orignal matrx $[P_{\mathbf{u}}]$.

1c. Geometrically, it is clear that the composition $P_{\mathbf{u}} \circ P_{\mathbf{w}} = 0$, collapsing the whole plane on the origin. This can be computed from the matrices using $\mathbf{u}\cdot\mathbf{w} = 0$.

2. The matrix of a mapping $\ell : \mathbb{R}^n \to \mathbb{R}^m$ has m rows and n columns (dimensions $m \times n$). The given mappings have matrices

$$[\,\ell_1\,] \;=\; [\,m_1 \;\; m_2\,], \qquad [\,\ell_2\,] \;=\; \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}.$$

Two functions can be composed only if the output of the second function is an acceptable input of the first function; thus $\ell_1 \circ \ell_1$ and $\ell_2 \circ \ell_2$ are not defined.

The allowable compositions are $\ell_1 \circ \ell_2 : \mathbb{R} \to \mathbb{R}$, with $\ell_1(\ell_2(t)) = (m_1a_1 + m_2a_2)\,t = (\mathbf{m}\cdot\mathbf{a})\,t$, and:

$$[\ell_2 \circ \ell_1] \;=\; \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}\cdot[\,m_1 \;\; m_2\,] \;=\; \begin{bmatrix} a_1m_1 & a_1m_2 \\ a_2m_1 & a_2m_2 \end{bmatrix}.$$

This is not very nice geometrically, but it can be thought of as projecting onto the direction $\mathbf{m} = (m_1, m_2)$, dilating by a factor of $|\mathbf{a}|\,|\mathbf{m}|$, then rotating the direction $\mathbf{m}$ to the direction $\mathbf{a}$.

3. The relation $\mathrm{Rot}_{\alpha+\beta} = \mathrm{Rot}_\alpha \circ \mathrm{Rot}_\beta$, translates into the matrix equations:

$$\begin{aligned}
\begin{bmatrix} \cos(\alpha{+}\beta) & -\sin(\alpha{+}\beta) \\ \sin(\alpha{+}\beta) & \cos(\alpha{+}\beta) \end{bmatrix} &= [\mathrm{Rot}_{\alpha+\beta}] \;=\; [\mathrm{Rot}_\alpha \circ \mathrm{Rot}_\beta] \;=\; [\mathrm{Rot}_\alpha]\cdot[\mathrm{Rot}_\beta] \\
&= \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix}\begin{bmatrix} \cos(\beta) & -\sin(\beta) \\ \sin(\beta) & \cos(\beta) \end{bmatrix} \\
&= \begin{bmatrix} \cos(\alpha)\cos(\beta){-}\sin(\alpha)\sin(\beta) & -\cos(\alpha)\sin(\beta){-}\sin(\alpha)\cos(\beta) \\ \sin(\alpha)\cos(\beta){+}\cos(\alpha)\sin(\beta) & -\sin(\alpha)\sin(\beta){+}\cos(\alpha)\cos(\beta) \end{bmatrix}
\end{aligned}$$

Equating entries in the first and last matrices gives the Angle Addition Formulas.