Math 254H **Recitation 1** **Aug 29, 1019**

Prove the following propositions of plane geometry by translating them into vector algebra. Do not use trigonometry or rely on known geometric results, only basic definitions.

1. The centroid of a triangle lies on each median, $\frac{2}{3}$ of the way from each vertex to the midpoint of the opposite side. (Thus, the centroid is the intersection point of the three medians.)

2. Given a triangle, the midpoints of its sides are by definition the vertices of the *medial triangle*. Show that the medial triangle has the same centroid as the original triangle.

3. The medial triangle is similar to the original triangle, with sides parallel to and half the length of the original sides.

4. Let $ABCD$ be a pallelogram with $AB||CD$ and $AC||BD$. Then $ABCD$ is a rhombus (having all side-lengths equal) if and only if the two diagonals are perpendicular to each other.

5. If a triangle ABC is inscribed in a circle with center O, then the orthogonal bisectors of the three sides meet at O.

6. If an angle is inscribed in a circle with one side being a diameter, the corresponding central angle is twice as large as the inscribed angle. *Hint:* Show that the inscribed angle is equal to the central angle between the diameter and the midpoint of the secant cutting across the inscribed angle.

7. If a circle with center O has an inscribed segment AB with midpoint M, then AB is perpendicular to OM.

8. If a circle has inscribed segments AB and CD intersecting at P, then:

$$|AP| \cdot |BP| = |CP| \cdot |DP| \,.$$

   Hint: Start with the intersection point $P = \vec{p}$, and let $A = \vec{p}+\vec{u}$, $B = \vec{p}-t\vec{u}$, $C = \vec{p}+\vec{v}$, $D = \vec{p}-s\vec{v}$, satisfying $|A|^2 = |B|^2 = |C|^2 = |D|^2 = r$. Solve for t, s in terms of $\vec{p}, \vec{u}, \vec{v}$, and evaluate $t|\vec{u}|^2$, $s|\vec{v}|^2$.

9. In a circle, an inscribed angle is half the size of the corresponding central angle.

10. (a) Consider a circle with center O, and a line that cuts the circle in segment QR with midpoint M. Then radius $OM \perp QR$.
    (b) A line tangent to the circle at M is perpendicular to radius OM.
    (c) From a point P outside a circle, if the two tangent lines touch the circle at Q and R, then $|PQ| = |PR|$.

11. Dandelin spheres (exposition for fun)