Additional Content for Vector Calculus

Sixth Edition

Jerrold E. Marsden
California Institute of Technology

Anthony Tromba
University of California, Santa Cruz

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Contents

Preface

The Structure of this Supplement. This Internet Supplement is intended to be used with the 6th Edition of our text *Vector Calculus.* It contains supplementary material that gives further information on various topics in Vector Calculus, including different applications and also technical proofs that were omitted from the main text.

The supplement is intended for students who wish to gain a deeper understanding, usually by self study, of the material—both for the theory as well as the applications.

Corrections and Website. A list of corrections and suggestions concerning the text and instructors guide are available available on the book's website:

http://www.whfreeman.com/MarsdenVC6e

Please send any new corrections you may find to one of us.

More Websites. There is of course a huge number of websites that contain a wealth of information. Here are a few sample sites that are relevant for the book:

1. For spherical geometry in Figure 8.2.13 of the main text, see http://torus.math.uiuc.edu/jms/java/dragsphere/ which gives a nice JAVA applet for parallel transport on the sphere.

2. For further information on the Sunshine formula (see §4.1C of this internet supplement), see http://www.math.niu.edu/~rusin/uses-math/position.sun/

3. For the Genesis Orbit shown in Figure 4.1.11, see
   http://genesismission.jpl.nasa.gov/

4. For more on Newton, see for instance,
   http://scienceworld.wolfram.com/biography/Newton.html
   and for Feynman, see
   http://www.feynman.com/

5. For surface integrals using Mathematica Notebooks, see
   http://www.math.umd.edu/~jmr/241/surfint.htm

Practice Final Examination. Also on the Book Companion Web Site, students will find a Practice Final Examination that covers topics in the whole book, complete with solutions. We recommend, if you wish to practice your skills, that you allow yourself 3 hours to take the exam and then self-mark it, keeping in mind that there is often more than one way to approach a problem.

Acknowledgements. As with the main text, the student guide, and the Instructors Manual, we are very grateful to the readers of earlier editions of the book for providing valuable advice and pointing out places where the text can be improved. For this internet supplement, we are especially grateful to Alan Weinstein for his collaboration in writing the supplement on the sunshine formula (see the supplement to Chapter 4) and for making a variety of other interesting and useful remarks. We thank a number of readers for their helpful comments on this supplement, including Brian Bradie, Dave Rusin and Paulo Sousa.

We send everyone who uses this supplement and the book our best regards and hope that you will enjoy your studies of vector calculus and that you will benefit (both intellectually and practically) from it.

Anthony Tromba (tromba@math.ucsc.edu)
Department of Mathematics
University of California
Santa Cruz, CA 95064

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Differentiation

In the first edition of *Principia* Newton admitted that Leibniz was in possession of a similar method (of tangents) but in the third edition of 1726, following the bitter quarrel between adherents of the two men concerning the independence and priority of the discovery of the calculus, Newton deleted the reference to the calculus of Leibniz. It is now fairly clear that Newton's discovery antedated that of Leibniz by about ten years, but that the discovery by Leibniz was independent of that of Newton. Moreover, Leibniz is entitled to priority of publication, for he printed an account of his calculus in 1684 in the *Acta Eruditorum*, a sort of "scientific monthly" that had been established only two years before.

Carl B. Boyer
A History of Mathematics

§2.7 Some Technical Differentiation Theorems

In this section we examine the mathematical foundations of differential calculus in further detail and supply some of the proofs omitted from §§2.2, 2.3, and 2.5.

Limit Theorems. We shall begin by supplying the proofs of the limit theorems presented in §2.2 (the theorem numbering in this section corresponds to that in Chapter 2). We first recall the definition of a limit.[1]

Definition of Limit. *Let $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$ where A is open. Let $\mathbf{x}_0$ be in A or be a boundary point of A, and let N be a neighborhood of $\mathbf{b} \in \mathbb{R}^m$. We say f is* ***eventually*** *in N as $\mathbf{x}$* ***approaches*** *$\mathbf{x}_0$ if there exists a neighborhood U of $\mathbf{x}_0$ such that $\mathbf{x} \neq \mathbf{x}_0, \mathbf{x} \in U$, and $\mathbf{x} \in A$ implies $f(\mathbf{x}) \in N$. We say $f(\mathbf{x})$* ***approaches*** *$\mathbf{b}$ as $\mathbf{x}$ approaches $\mathbf{x}_0$, or, in symbols,*

$$\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = \mathbf{b} \quad \textit{or} \quad f(\mathbf{x}) \to \mathbf{b} \quad \textit{as} \quad \mathbf{x} \to \mathbf{x}_0,$$

when, given any *neighborhood N of $\mathbf{b}$, f is eventually in N as $\mathbf{x}$ approaches $\mathbf{x}_0$. If, as $\mathbf{x}$ approaches $\mathbf{x}_0$, the values $f(\mathbf{x})$ do not get close to any particular number, we say that $\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x})$* ***does not exist****.*

Let us first establish that this definition is equivalent to the ε-δ formulation of limits. The following result was stated in §2.2.

Theorem 6. *Let $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$ and let $\mathbf{x}_0$ be in A or be a boundary point of A. Then $\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = \mathbf{b}$ if and only if for every number $\varepsilon > 0$ there is a $\delta > 0$ such that for any $\mathbf{x} \in A$ satisfying $0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta$, we have $\|f(\mathbf{x}) - \mathbf{b}\| < \varepsilon$.*

Proof. First let us assume that $\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = \mathbf{b}$. Let $\varepsilon > 0$ be given, and consider the ε neighborhood $N = D_\varepsilon(\mathbf{b})$, the ball or disk of radius ε with center $\mathbf{b}$. By the definition of a limit, f is eventually in $D_\varepsilon(\mathbf{b})$, as $\mathbf{x}$ approaches $\mathbf{x}_0$, which means there is a neighborhood U of $\mathbf{x}_0$ such that $f(\mathbf{x}) \in D_\varepsilon(\mathbf{b})$ if $\mathbf{x} \in U$, $\mathbf{x} \in A$, and $\mathbf{x} \neq \mathbf{x}_0$. Now since U is open and $\mathbf{x}_0 \in U$, there is a $\delta > 0$ such that $D_\delta(\mathbf{x}_0) \subset U$. Consequently, $0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta$ and $\mathbf{x} \in A$ implies $\mathbf{x} \in D_\delta(\mathbf{x}_0) \subset U$. Thus $f(\mathbf{x}) \in D_\varepsilon(\mathbf{b})$, which means that $\|f(\mathbf{x}) - \mathbf{b}\| < \varepsilon$. This is the ε-δ assertion we wanted to prove.

We now prove the converse. Assume that for every $\varepsilon > 0$ there is a $\delta > 0$ such that $0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta$ and $\mathbf{x} \in A$ implies $\|f(\mathbf{x}) - \mathbf{b}\| < \varepsilon$. Let N be a neighborhood of $\mathbf{b}$. We have to show that f is eventually in N as $\mathbf{x} \to \mathbf{x}_0$; that is, we must find an open set $U \subset \mathbb{R}^n$ such that $\mathbf{x} \in U, \mathbf{x} \in A$, and $\mathbf{x} \neq \mathbf{x}_0$ implies $f(\mathbf{x}) \in N$. Now since N is open, there is an $\varepsilon > 0$ such that $D_\varepsilon(\mathbf{b}) \subset N$. If we choose $U = D_\delta(\mathbf{x})$ (according to our assumption), then $\mathbf{x} \in U$, $\mathbf{x} \in A$ and $\mathbf{x} \neq \mathbf{x}_0$ means $\|f(\mathbf{x}) - \mathbf{b}\| < \varepsilon$, that is $f(\mathbf{x}) \in D_\varepsilon(\mathbf{b}) \subset N$. ■

Properties of Limits. The following result was also stated in §2.2. Now we are in a position to provide the proof.

[1] For those interested in a different pedagogical approach to limits and the derivative, we recommend *Calculus Unlimited* by J. Marsden and A. Weinstein. It is freely available on the Vector Calculus website given in the Preface.

Theorem 2. Uniqueness of Limits. *If*

$$\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = \mathbf{b}_1 \quad \textit{and} \quad \lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = \mathbf{b}_2,$$

then $\mathbf{b}_1 = \mathbf{b}_2$.

Proof. It is convenient to use the ε-δ formulation of Theorem 6. Suppose $f(\mathbf{x}) \to \mathbf{b}_1$ and $f(\mathbf{x}) \to \mathbf{b}_2$ as $\mathbf{x} \to \mathbf{x}_0$. Given $\varepsilon > 0$, we can, by assumption, find $\delta_1 > 0$ such that if $\mathbf{x} \in A$ and $0 < \|\mathbf{x}-\mathbf{x}_0\| < \delta_2$, then $\|f(\mathbf{x})-\mathbf{b}_1\| < \varepsilon$, and similarly, we can find $\delta_1 > 0$ such that $0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta_2$ implies $\|f(\mathbf{x}) - \mathbf{b}_2\| < \varepsilon$. Let δ be the smaller of δ_1 and δ_2. Choose $\mathbf{x}$ such that $0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta$ and $\mathbf{x} \in A$. Such $\mathbf{x}$'s exist, because $\mathbf{x}_0$ is in A or is a boundary point of A. Thus, using the triangle inequality,

$$\begin{aligned}\|\mathbf{b}_1 - \mathbf{b}_2\| &= \|(\mathbf{b}_1 - f(\mathbf{x})) + (f(\mathbf{x}) - \mathbf{b}_2)\| \\ &\le \|\mathbf{b}_1 - f(\mathbf{x})\| + \|f(\mathbf{x}) - \mathbf{b}_2\| < \varepsilon + \varepsilon = 2\varepsilon.\end{aligned}$$

Thus for *every* $\varepsilon > 0$, $\|\mathbf{b}_1 - \mathbf{b}_2\| < 2\varepsilon$. Hence $\mathbf{b}_1 = \mathbf{b}_2$, for if $\mathbf{b}_1 \neq \mathbf{b}_2$ we could let $\varepsilon = \|\mathbf{b}_1 - \mathbf{b}_2\|/2 > 0$ and we would have $\|\mathbf{b}_1 - \mathbf{b}_2\| < \|\mathbf{b}_1 - \mathbf{b}_2\|$, an impossibility. ■

The following result was also stated without proof in §2.2.

Theorem 3. Properties of Limits. *Let* $f : A \subset \mathbb{R}^n \to \mathbb{R}^m, g : A \subset \mathbb{R}^n \to \mathbb{R}^m, \mathbf{x}_0$ *be in* A *or be a boundary point of* $A, \mathbf{b} \in \mathbb{R}^m$, *and* $c \in \mathbb{R}$; *the following assertions than hold:*

(i) *If* $\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = \mathbf{b}$, *then* $\lim_{\mathbf{x}\to\mathbf{x}_0} cf(\mathbf{x}) = c\mathbf{b}$, *where* $cf : A \to \mathbb{R}^m$ *is defined by* $\mathbf{x} \longmapsto c(f(\mathbf{x}))$.

(ii) *If* $\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) - \mathbf{b}_1$ *and* $\lim_{\mathbf{x}\to\mathbf{x}_0} g(\mathbf{x}) = \mathbf{b}_2$, *then*

$$\lim_{\mathbf{x}\to\mathbf{x}_0} (f + g)(\mathbf{x}) = \mathbf{b}_1 + \mathbf{b}_2,$$

where $(f + g) : A \to \mathbb{R}^m$ *is defined by* $\mathbf{x} \longmapsto f(\mathbf{x}) + g(\mathbf{x})$.

(iii) *If* $m = 1, \lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = b_1$, *and* $\lim_{\mathbf{x}\to\mathbf{x}_0} g(\mathbf{x}) = b_2$, *then*

$$\lim_{\mathbf{x}\to\mathbf{x}_0} (fg)(\mathbf{x}) = b_1 b_2,$$

where $(fg) : A \to \mathbb{R}$ *is defined by* $\mathbf{x} \longmapsto f(\mathbf{x})g(\mathbf{x})$.

(iv) *If* $m = 1, \lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = b \neq 0$, *and* $f(\mathbf{x}) \neq 0$ *for all* $\mathbf{x} \in A$, *then*

$$\lim_{\mathbf{x}\to\mathbf{x}_0} \frac{1}{f} = \frac{1}{b},$$

where $1/f : A \to \mathbb{R}$ *is defined by* $\mathbf{x} \longmapsto 1/f(\mathbf{x})$.

(v) *If* $f(\mathbf{x}) = (f_1(\mathbf{x}), \ldots, f_m(\mathbf{x}))$*, where* $f_i : A \to \mathbb{R}, i = 1, \ldots, m$*, are the component functions of* f*, then* $\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = \mathbf{b} = (b_1, \ldots, b_m)$ *if and only if* $\lim_{\mathbf{x}\to\mathbf{x}_0} f_i(\mathbf{x}) = b_i$ *for each* $i = 1, \ldots, m$*.*

Proof. We shall illustrate the technique of proof by proving assertions (i) and (ii). The proofs of the other assertions are only a bit more complicated and may be supplied by the reader. In each case, the ε-δ formulation of Theorem 6 is probably the most convenient approach.

To prove rule (i), let $\varepsilon > 0$ be given; we must produce a number $\delta > 0$ such that the inequality $\|cf(\mathbf{x}) - c\mathbf{b}\| < \varepsilon$ holds if $0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta$. If $c = 0$, any δ will do, so we can suppose $c \neq 0$. Let $\varepsilon' = \varepsilon/|c|$; from the definition of limit, there is a δ with the property that $0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta$ implies $\|f(\mathbf{x}) - \mathbf{b}\| < \varepsilon' = \varepsilon/|c|$. Thus $0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta$ implies $\|cf(\mathbf{x}) - c\mathbf{b}\| = |c|\|f(\mathbf{x}) - \mathbf{b}\|| < \varepsilon$, which proves rule (i).

To prove rule (ii), let $\varepsilon > 0$ be given again. Choose $\delta_1 > 0$ such that $0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta_1$ implies $\|f(\mathbf{x}) - \mathbf{b}_1\| < \varepsilon/2$. Similarly, choose $\delta_2 > 0$ such that $0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta_2$ implies $\|g(\mathbf{x}) - \mathbf{b}_2\| < \varepsilon/2$. Let δ be the lesser of δ_1 and δ_2. Then $0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta$ implies

$$\|f(\mathbf{x}) + g(\mathbf{x}) - \mathbf{b}_1 - \mathbf{b}_2\| \leq \|f(\mathbf{x}) - \mathbf{b}_1\| + \|g(\mathbf{x}) - \mathbf{b}_2\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus, we have proved that $(f + g)(\mathbf{x}) \to \mathbf{b}_1 + \mathbf{b}_2$ as $\mathbf{x} \to \mathbf{x}_0$. ■

Example 1. Find the following limit if it exists:

$$\lim_{(x,y)\to(0,0)} \left(\frac{x^3 - y^3}{x^2 + y^2}\right).$$

Solution. Since

$$0 \leq \left|\frac{x^3 - y^3}{x^2 + y^2}\right| \leq \frac{|x|x^2 + |y|y^2}{x^2 + y^2} \leq \frac{(|x| + |y|)(x^2 + y^2)}{x^2 + y^2} = |x| + |y|,$$

we find that

$$\lim_{(x,y)\to(0,0)} \frac{x^3 - y^3}{x^2 + y^2} = 0.$$ ♦

Continuity. Now that we have the limit theorems available, we can use this to study continuity; we start with the basic definition of continuity of a function.

Definition. *Let* $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$ *be a given function with domain* A*. Let* $\mathbf{x}_0 \in A$*. We say* f *is* ***continuous at*** $\mathbf{x}_0$ *if and only if*

$$\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = f(\mathbf{x}_0).$$

If we say that f *is* ***continuous****, we shall mean that* f *is continuous at each point* $\mathbf{x}_0$ *of* A*.*

From Theorem 6, we get the ε-δ criterion for continuity.

Theorem 7. *A mapping $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$ is continuous at $\mathbf{x}_0 \in A$ if and only if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that*

$$\mathbf{x} \in A \;\; \textit{and} \;\; \|\mathbf{x} - \mathbf{x}_0\| < \delta \;\; \textit{implies} \;\; \|f(\mathbf{x}) - f(\mathbf{x}_0)\| < \varepsilon.$$

One of the properties of continuous functions stated without proof in §2.2 was the following:

Theorem 5. Continuity of Compositions. *Let $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$ and let $g : B \subset \mathbb{R}^m \to \mathbb{R}^p$. Suppose $f(A) \subset B$ so that $g \circ f$ is defined on A. If f is continuous at $\mathbf{x}_0 \in A$ and g is continuous at $\mathbf{y}_0 = f(\mathbf{x}_0)$, then $g \circ f$ is continuous at $\mathbf{x}_0$.*

Proof. We use the ε-δ criterion for continuity. Thus, given $\varepsilon > 0$, we must find $\delta > 0$ such that for $\mathbf{x} \in A$.

$$\|\mathbf{x} - \mathbf{x}_0\| < \delta \;\; \text{implies} \;\; \|(g \circ f)(\mathbf{x}) - (g \circ f)(\mathbf{x}_0)\| < \varepsilon.$$

Since g is continuous at $f(\mathbf{x}_0) = \mathbf{y}_0 \in B$, there is a $\gamma > 0$ such that for $\mathbf{y} \in B$,

$$\|\mathbf{y} - \mathbf{y}_0\| < \gamma \;\; \text{implies} \;\; \|g(\mathbf{y}) - g(f(\mathbf{x}_0))\| < \varepsilon.$$

Since f is continuous $\mathbf{x}_0 \in A$, there is, for this γ, a $\delta > 0$ such that for $\mathbf{x} \in A$,

$$\|\mathbf{x} - \mathbf{x}_0\| < \delta \;\; \text{implies} \;\; \|f(\mathbf{x}) - f(\mathbf{x}_0)\| < \gamma,$$

which in turn implies

$$\|g(f(\mathbf{x})) - g(f(\mathbf{x}_0))\| < \varepsilon,$$

which is the desired conclusion. ■

Differentiability. The exposition in §2.3 was simplified by assuming, as part of the definition of $\mathbf{D}f(\mathbf{x}_0)$, that the partial derivatives of f existed. Our next objective is to show that *this assumption can be omitted*. Let us begin by *redefining* "differentiable." Theorem 15 below will show that the new definition is equivalent to the old one.

Definition. *Let U be an open set in $\mathbb{R}^n$ and let $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$ be a given function. We say that f is **differentiable** at $\mathbf{x}_0 \in U$ if and only if there exists an $m \times n$ matrix $\mathbf{T}$ such that*

$$\lim_{\mathbf{x} \to \mathbf{x}_0} \frac{\|f(\mathbf{x}) - f(\mathbf{x}_0) - \mathbf{T}(\mathbf{x} - \mathbf{x}_0)\|}{\|\mathbf{x} - \mathbf{x}_0\|} = 0. \tag{1}$$

We call $\mathbf{T}$ the ***derivative*** of f at $\mathbf{x}_0$ and denote it by $\mathbf{D}f(\mathbf{x}_0)$. In matrix notation, $\mathbf{T}(\mathbf{x}-\mathbf{x}_0)$ stands for

$$\begin{bmatrix} T_{11} & T_{12} & \dots & T_{1n} \\ T_{21} & T_{22} & \dots & T_{2_n} \\ \vdots & \vdots & & \vdots \\ T_{m1} & T_{m2} & \dots & T_{mn} \end{bmatrix} \begin{bmatrix} x_1 - x_{01} \\ \vdots \\ x_n - x_{0n} \end{bmatrix}.$$

where $\mathbf{x} = (x_1, \dots, x_n)$, $\mathbf{x}_0 = (x_{01}, \dots, x_{0n})$, and where the matrix entries of $\mathbf{T}$ are denoted $[T_{ij}]$. Sometimes we write $\mathbf{T}(\mathbf{y})$ as $\mathbf{T} \cdot \mathbf{y}$ or just $\mathbf{Ty}$ for the product of the matrix $\mathbf{T}$ with the column vector $\mathbf{y}$.

Condition (1) can be rewritten as

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{\|f(\mathbf{x}_0+\mathbf{h}) - f(\mathbf{x}_0) - \mathbf{Th}\|}{\|\mathbf{h}\|} = 0 \tag{2}$$

as we see by letting $\mathbf{h} = \mathbf{x}-\mathbf{x}_0$. Written in terms of ε-δ notation, equation (2) says that for every $\varepsilon > 0$ there is a $\delta > 0$ such that $0 < \|\mathbf{h}\| < \delta$ implies

$$\frac{\|f(\mathbf{x}_0+\mathbf{h}) - f(\mathbf{x}_0) - \mathbf{Th}\|}{\|\mathbf{h}\|} < \varepsilon,$$

or, in other words,

$$\|f(\mathbf{x}_0+\mathbf{h}) - f(\mathbf{x}_0) - \mathbf{Th}\| < \varepsilon\|\mathbf{h}\|.$$

Notice that because U is open, as long as δ is small enough, $\|\mathbf{h}\| < \delta$ implies $\mathbf{x}_0 + \mathbf{h} \in U$.

Our first task is to show that the matrix $\mathbf{T}$ is necessarily the matrix of partial derivatives, and hence that this abstract definition agrees with the definition of differentiability given in §2.3.

Theorem 15. *Suppose $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $\mathbf{x}_0 \in \mathbb{R}^n$. Then all the partial derivatives of f exist at the point $\mathbf{x}_0$ and the $m \times n$ matrix $\mathbf{T}$ has entries given by*

$$[T_{ij}] = \left[\frac{\partial f_i}{\partial x_j}\right],$$

that is,

$$\mathbf{T} = \mathbf{D}f(\mathbf{x}_0) = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \cdots & \dfrac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \dfrac{\partial f_m}{\partial x_1} & \cdots & \dfrac{\partial f_m}{\partial x_n} \end{bmatrix},$$

where $\partial f_i / \partial x_j$ is evaluated at $\mathbf{x}_0$. In particular, this implies that $\mathbf{T}$ is uniquely determined; that is, there is no other matrix satisfying condition (1).

Proof. By Theorem 3(v), condition (2) is the same as

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{|f_i(\mathbf{x}_0+\mathbf{h}) - f_i(\mathbf{x}_0) - (\mathbf{Th})_i|}{\|\mathbf{h}\|} = 0, \quad 1 \le i \le m.$$

Here $(\mathbf{Th}_i)$ stands for the ith component of the column vector $\mathbf{Th}$. Now let $\mathbf{h} = a\mathbf{e}_j = (0, \ldots, a, \ldots, 0)$, which has the number a in the jth slot and zeros elsewhere. We get

$$\lim_{a\to 0} \frac{|f_i(\mathbf{x}_0+a\mathbf{e}_j) - f_i(\mathbf{x}_0) - a(\mathbf{Te}_j)_i|}{|a|} = 0,$$

or, in other words,

$$\lim_{a\to 0} \left| \frac{f_i(\mathbf{x}_0+a\mathbf{e}_j) - f_i(\mathbf{x}_0)}{a} - (\mathbf{Te}_j)_i \right| = 0,$$

so that

$$\lim_{a\to 0} \frac{f_i(\mathbf{x}_0+a\mathbf{e}_j) - f_i(\mathbf{x}_0)}{a} = (\mathbf{Te}_j)_i.$$

But this limit is nothing more than the partial derivative $\partial f_i/\partial x_j$ evaluated at the point $\mathbf{x}_0$. Thus, we have proved that $\partial f_i/\partial x_j$ exists and equals $(\mathbf{Te}_j)_i$. But $(\mathbf{Te}_j)_i = T_{ij}$ (see §1.5 of the main text), and so the theorem follows. ■

Differentiability and Continuity. Our next task is to show that differentiability implies continuity.

Theorem 8. *Let $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$ be differentiable at $\mathbf{x}_0$. Then f is continuous at $\mathbf{x}_0$, and furthermore, $\|f(\mathbf{x}) - f(\mathbf{x}_0)\| < M_1\|\mathbf{x} - \mathbf{x}_0\|$ for some constant M_1 and $\mathbf{x}$ near $\mathbf{x}_0, \mathbf{x} \ne \mathbf{x}_0$.*

Proof. We shall use the result of Exercise 2 at the end of this section, namely that,

$$\|\mathbf{D}f(\mathbf{x}_0)\cdot\mathbf{h}\| \le M\|\mathbf{h}\|,$$

where M is the square root of the sum of the squares of the matrix elements in $\mathbf{D}f(\mathbf{x}_0)$.

Choose $\varepsilon = 1$. Then by the definition of the derivative (see formula (2)) there is a $\delta_1 > 0$ such that $0 < \|\mathbf{h}\| < \delta_1$ implies

$$\|f(\mathbf{x}_0+\mathbf{h}) - f(\mathbf{x}_0) - \mathbf{D}f(\mathbf{x}_0)\cdot\mathbf{h}\| < \varepsilon\|\mathbf{h}\| = \|\mathbf{h}\|.$$

If $\|\mathbf{h}\| < \delta_1$, then using the triangle inequality,

$$\begin{aligned}\|f(\mathbf{x}_0+\mathbf{h}) - f(\mathbf{x}_0)\| &= \|f(\mathbf{x}_0+\mathbf{h}) - f(\mathbf{x}_0) - \mathbf{D}f(\mathbf{x}_0)\cdot\mathbf{h} + \mathbf{D}f(\mathbf{x}_0)\cdot\mathbf{h}\| \\ &\le \|f(\mathbf{x}_0+\mathbf{h}) - f(\mathbf{x}_0) - \mathbf{D}f(\mathbf{x}_0)\cdot\mathbf{h}\| + \|\mathbf{D}f(\mathbf{x}_0)\cdot\mathbf{h}\| \\ &< \|\mathbf{h}\| + M\|\mathbf{h}\| = (1+M)\|\mathbf{h}\|.\end{aligned}$$

Setting $\mathbf{x} = \mathbf{x}_0 + \mathbf{h}$ and $M_1 = 1 + M$, we get the second assertion of the theorem.

Now let ε' be any positive number, and let δ be the smaller of the two positive numbers δ_1 and $\varepsilon'/(1+M)$. Then $\|\mathbf{h}\| < \delta$ implies

$$\|f(\mathbf{x}_0 + \mathbf{h}) - f(\mathbf{x}_0)\| < (1+M)\frac{\varepsilon'}{1+M} = \varepsilon',$$

which proves that (see Exercise 15 at the end of this section)

$$\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = \lim_{\mathbf{h}\to\mathbf{0}} f(\mathbf{x}_0 + \mathbf{h}) = f(\mathbf{x}_0),$$

so that f is continuous at $\mathbf{x}_0$. ■

Criterion for Differentiability. We asserted in §2.3 that an important criterion for differentiability is that the partial derivatives exist and are continuous. We now are able to prove this.

Theorem 9. *Let $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$. Suppose the partial derivatives $\partial f_i/\partial x_j$ of f all exist and are continuous in some neighborhood of a point $\mathbf{x} \in U$. Then f is differentiable at $\mathbf{x}$.*

Proof. In this proof we are going to use the mean value theorem from one-variable calculus—see §2.5 of the main text for the statement. To simplify the exposition, we shall only consider the case $m = 1$, that is, $f : U \subset \mathbb{R}^n \to \mathbb{R}$, leaving the general case to the reader (this is readily supplied knowing the techniques from the proof of Theorem 15, above).

According to the definition of the derivative, our objective is to show that

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{\left| f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x}) - \sum_{i=1}^{n} \left[\frac{\partial f}{\partial x_i}(\mathbf{x})\right] h_i \right|}{\|\mathbf{h}\|} = 0.$$

Write

$$\begin{aligned}
&f(x_1+h_1, \dots, x_n+h_n) - f(x_1, \dots, x_n)\\
&= f(x_1+h_1, \dots, x_n+h_n) - f(x_1, x_2+h_2, \dots, x_n+h_n)\\
&\quad + f(x_1, x_2+h_2, \dots, x_n+h_n) - f(x_1, x_2, x_3+h_3, \dots, x_n+h) + \dots\\
&\quad + f(x_1, \dots, x_{n-1}+h_{n-1}, x_n+h_n) - f(x_1, \dots, x_{n-1}, x_n+h_n)\\
&\quad + f(x_1, \dots, x_{n-1}, x_n+h_n) - f(x_1, \dots, x_n).
\end{aligned}$$

This is called a ***telescoping sum***, since each term cancels with the succeeding or preceding one, except the first and the last. By the mean value theorem, this expression may be written as

$$f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x}) = \left[\frac{\partial f}{\partial x_1}(\mathbf{y}_1)\right] h_1 + \left[\frac{\partial f}{\partial x_2}(\mathbf{y}_2)\right] h_2 + \dots + \left[\frac{\partial f}{\partial x_n}(\mathbf{y}_2)\right] h_n,$$

where $\mathbf{y}_1 = (c_1, x_2 + h_2, \ldots, x_n + h_n)$ with c_1 lying between x_1 and $x_1 + h_1$; $\mathbf{y}_2 = (x_1, c_2, x_3 + h_3, \ldots, x_n + h_n)$ with c_2 lying between x_2 and $x_2 + h_2$; and $\mathbf{y}_n = (x_1, \ldots, x_{n-1}, c_n)$ where c_n lies between x_n and $x_n + h_n$. Thus, we can write

$$\begin{aligned}&\left| f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x}) - \sum_{i=1}^{n} \left[\frac{\partial f}{\partial x_i}(\mathbf{x})\right] h_i \right| \\ &\quad = \left| \left(\frac{\partial f}{\partial x_1}(\mathbf{y}_1) - \frac{\partial f}{\partial x_1}(\mathbf{x})\right) h_1 + \ldots + \left(\frac{\partial f}{\partial x_n}(\mathbf{y}_n) - \frac{\partial f}{\partial x_n}(\mathbf{x})\right) h_n \right|.\end{aligned}$$

By the triangle inequality, this expression is less than or equal to

$$\begin{aligned}&\left|\frac{\partial f}{\partial x_1}(\mathbf{y}_1) - \frac{\partial f}{\partial x_1}(\mathbf{x})\right| |h_1| + \ldots + \left|\frac{\partial f}{\partial x_n}(\mathbf{y}_n) - \frac{\partial f}{\partial x_n}(\mathbf{x})\right| |h_n| \\ &\quad \le \left\{ \left|\frac{\partial f}{\partial x_1}(\mathbf{y}_1) - \frac{\partial f}{\partial x_1}(\mathbf{x})\right| + \ldots + \left|\frac{\partial f}{\partial x_n}(\mathbf{y}_n) - \frac{\partial f}{\partial x_n}(\mathbf{x})\right| \right\} \|\mathbf{h}\|.\end{aligned}$$

since $|h_i| \le \|\mathbf{h}\|$ for all i. Thus, we have proved that

$$\begin{aligned}&\frac{\left| f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x}) - \sum_{i=1}^{n} \left[\frac{\partial f}{\partial x_i}(\mathbf{x})\right] h_i \right|}{\|\mathbf{h}\|} \\ &\qquad \le \left|\frac{\partial f}{\partial x_1}(\mathbf{y}_1) - \frac{\partial f}{\partial x_1}(\mathbf{x})\right| + \ldots + \left|\frac{\partial f}{\partial x_n}(\mathbf{y}_n) - \frac{\partial f}{\partial x_n}(\mathbf{x})\right|.\end{aligned}$$

But since the partial derivatives are continuous by assumption, the right side approaches 0 as $\mathbf{h} \to \mathbf{0}$ so that the left side approaches 0 as well. ■

Here is an interesting point about this proof: while the proof requires that the partial derivatives of f exist in a neighborhood of $\mathbf{x}$, it only requires continuity of the partial derivatives *at the point* $\mathbf{x}$. This proof also uses the mean value theorem for functions of one variable and so it is important to notice that the mean value theorem from single variable calculus for a continuous function $f : [a, b] \to \mathbb{R}$ does not require continuity of the derivative, only its existence on the open interval (a, b). [2]

Chain Rule. As explained in §2.5, the Chain Rule is right up there with the most important results in differential calculus. We are now in a position to give a careful proof.

Theorem 11: Chain Rule. *Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be open. Let $g : U \subset \mathbb{R}^n \to \mathbb{R}^m$ and $f : V \subset \mathbb{R}^m \to \mathbb{R}^p$ be given functions such that g*

---

[2]For the precise statement of the mean value theorem, see any good single variable Calculus textbook or a basic book on real analysis, such as *Elementary Classical Analysis*, Second Edition, by Marsden and Hoffman, W.H. Freeman and Co., 1993.

maps U into V, so that $f \circ g$ is defined. Suppose g is differentiable at $\mathbf{x}_0$ and f is differentiable at $\mathbf{y}_0 = g(\mathbf{x}_0)$. Then $f \circ g$ is differentiable at $\mathbf{x}_0$ and

$$\mathbf{D}(f \circ g)(\mathbf{x}_0) = \mathbf{D}f(\mathbf{y}_0)\mathbf{D}g(\mathbf{x}_0).$$

Proof. According to the definition of the derivative, we must verify that

$$\lim_{\mathbf{x}\to\mathbf{x}_0} \frac{\|f(g(\mathbf{x})) - f(g(\mathbf{x}_0)) - \mathbf{D}f(\mathbf{y}_0)\mathbf{D}g(\mathbf{x}_0)\cdot(\mathbf{x}-\mathbf{x}_0)\|}{\|\mathbf{x}-\mathbf{x}_0\|} = 0.$$

First rewrite the numerator and apply the triangle inequality as follows:

$$\begin{aligned} &\|f(g(\mathbf{x})) - f(g(\mathbf{x}_0)) - \mathbf{D}f(\mathbf{y}_0 \cdot (g(\mathbf{x}) - g(\mathbf{x}_0)) \\ &\qquad + \mathbf{D}f(\mathbf{y}_0)\cdot[g(\mathbf{x}) - g(\mathbf{x}_0) - \mathbf{D}g(\mathbf{x}_0)\cdot(\mathbf{x}-\mathbf{x}_0)]\| \\ &\quad \le \|f(g(\mathbf{x})) - f(g(\mathbf{x}_0)) - \mathbf{D}f(\mathbf{y}_0)\cdot(g(\mathbf{x}) - g(\mathbf{x}_0))\| \\ &\qquad + \|\mathbf{D}f(\mathbf{y}_0)\cdot[g(\mathbf{x}) - g(\mathbf{x}_0) - \mathbf{D}g(\mathbf{x}_0)\cdot(\mathbf{x}-\mathbf{x}_0])\|. \end{aligned} \tag{3}$$

As in the proof of Theorem 8, $\|\mathbf{D}f(\mathbf{y}_0)\cdot\mathbf{h}\| \le M\|\mathbf{h}\|$ for some constant M. Thus the right-hand side of inequality (3) is less than or equal to

$$\begin{aligned} &\|f(g(\mathbf{x})) - f(g(\mathbf{x}_0)) - \mathbf{D}f(\mathbf{y}_0)\cdot(g(\mathbf{x}) - g(\mathbf{x}_0))\| \\ &\qquad + M\|g(\mathbf{x}) - g(\mathbf{x}_0) - \mathbf{D}g(\mathbf{x}_0)\cdot(\mathbf{x}-\mathbf{x}_0)\|. \end{aligned} \tag{4}$$

Since g is differentiable at $\mathbf{x}_0$, given $\varepsilon > 0$, there is a $\delta_1 > 0$ such that $0 < \|\mathbf{x}-\mathbf{x}_0\| < \delta_1$ implies

$$\frac{\|g(\mathbf{x}) - g(\mathbf{x}_0) - \mathbf{D}g(\mathbf{x}_0)\cdot(\mathbf{x}-\mathbf{x}_0)\|}{\|\mathbf{x}-\mathbf{x}_0\|} < \frac{\varepsilon}{2M}.$$

This makes the second term in expression (4) less than $\varepsilon\|\mathbf{x}-\mathbf{x}_0\|/2$.

Let us turn to the first term in expression (4). By Theorem 8,

$$\|g(\mathbf{x}) - g(\mathbf{x}_0)\| < M_1\|\mathbf{x}-\mathbf{x}_0\|$$

for a constant M_1 if $\mathbf{x}$ is near $\mathbf{x}_0$, say $0 < \|\mathbf{x}-\mathbf{x}_0\| < \delta_2$. Now choose δ_3 such that $0 < \|\mathbf{y}-\mathbf{y}_0\| < \delta_3$ implies

$$\|f(\mathbf{y}) - f(\mathbf{y}_0) - \mathbf{D}f(\mathbf{y}_0)\cdot(\mathbf{y}-\mathbf{y}_0)\| < \frac{\varepsilon\|\mathbf{y}-\mathbf{y}_0\|}{2M_1}.$$

Since $\mathbf{y} = g(\mathbf{x})$ and $\mathbf{y}_0 = g(\mathbf{x}_0)$, $\|\mathbf{y}-\mathbf{y}_0\| < \delta_3$ if $\|\mathbf{x}-\mathbf{x}_0\| < \delta_3/M_1$ and $\|\mathbf{x}-\mathbf{x}_0\| < \delta_2$, and so

$$\begin{aligned} &\|f(g(\mathbf{x})) - f(g(\mathbf{x}_0)) - \mathbf{D}f(\mathbf{y}_0)\cdot(g(\mathbf{x}) - g(\mathbf{x}_0))\| \\ &\qquad \le \frac{\varepsilon\|g(\mathbf{x}) - g(\mathbf{x}_0)\|}{2M_1} < \frac{\varepsilon\|\mathbf{x}-\mathbf{x}_0\|}{2}. \end{aligned}$$

Thus if $\delta = \min(\delta_1, \delta_2, \delta_3/M_1)$, expression (4) is less than

$$\frac{\varepsilon\|\mathbf{x}-\mathbf{x}_0\|}{2} + \frac{\varepsilon\|\mathbf{x}-\mathbf{x}_0\|}{2} = \varepsilon\|\mathbf{x}-\mathbf{x}_0\|,$$

and so

$$\frac{\|f(g(\mathbf{x})) - f(g(\mathbf{x}_0)) - \mathbf{D}f(\mathbf{y}_0)\mathbf{D}g(\mathbf{x}_0)(\mathbf{x}-\mathbf{x}_0)\|}{\|\mathbf{x}-\mathbf{x}_0\|} < \varepsilon$$

for $0 < \|\mathbf{x}-\mathbf{x}_0\| < \delta$. This proves the theorem. ■

A Crinkled Function. The student has already met with a number of examples illustrating the above theorems. Let us consider one more of a more technical nature.

Example 2. Let

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0). \end{cases}$$

Is f differentiable at $(0,0)$? (See Figure 2.7.1.)

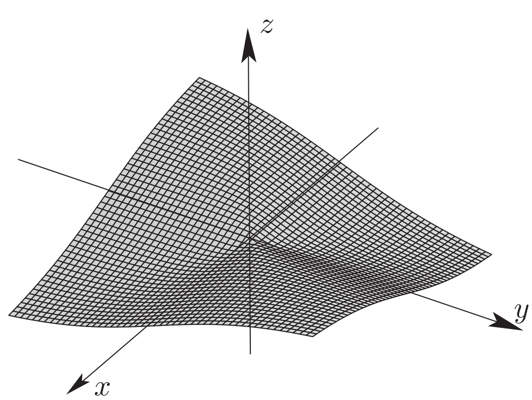

FIGURE 2.7.1. This function is not differentiable at $(0,0)$, because it is "crinkled."

Solution. We note that

$$\begin{aligned}\frac{\partial f}{\partial x}(0,0) &= \lim_{x\to 0} \frac{f(x,0)-f(0,0)}{x} \\ &= \lim_{x\to 0} \frac{(x\cdot 0)/\sqrt{x^2+0} - 0}{x} = \lim_{x\to 0} \frac{0-0}{x} = 0\end{aligned}$$

and similarly, $(\partial f/\partial y)(0,0) = 0$. Thus the partial derivatives exist at $(0,0)$. Also, if $(x,y) \neq (0,0)$, then

$$\frac{\partial f}{\partial x} = \frac{y\sqrt{x^2+y^2} - 2x(xy)/2\sqrt{x^2+y^2}}{x^2+y^2} = \frac{y}{\sqrt{x^2+y^2}} - \frac{x^2 y}{(x^2+y^2)^{3/2}},$$

which does not have a limit as $(x,y) \to (0,0)$. Different limits are obtained for different paths of approach, as can be seen by letting $x = My$. Thus the partial derivatives are *not continuous* at $(0,0)$, and so we cannot apply Theorem 9.

We might now try to show that f is not differentiable (f is continuous, however). If $\mathbf{D}f(0,0)$ existed, then by Theorem 15 it would have to be the zero matrix, since $\partial f/\partial x$ and $\partial f/\partial y$ are zero at $(0,0)$. Thus, by definition of differentiability, for any $\varepsilon > 0$ there would be a $\delta > 0$ such that $0 < \|(h_1,h_2)\| < \delta$ implies

$$\frac{|f(h_1,h_2) - f(0,0)|}{\|(h_1,h_2)\|} < \varepsilon$$

that is, $|f(h_1,h_2)| < \varepsilon\|(h_1,h_2)\|$, or $|h_1 h_2| < \varepsilon(h_1^2 + h_2^2)$. But if we choose $h_1 = h_2$, this reads $1/2 < \varepsilon$, which is untrue if we choose $\varepsilon \leq 1/2$. Hence, we conclude that f is *not differentiable* at $(0,0)$. ♦

Exercises[3]

1. Let $f(x,y,z) = (e^x, \cos y, \sin z)$. Compute $\mathbf{D}f$. In general, when will $\mathbf{D}f$ be a diagonal matrix?

2. (a) Let $\mathbf{A} : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with matrix $\{A_{ij}\}$ so that $\mathbf{A}\mathbf{x}$ has components $y_i = \sum_j A_{ij} x_j$. Let

$$M = \left(\sum_{ij} A_{ij}^2\right)^{1/2}.$$

Use the Cauchy-Schwarz inequality to prove that $\|\mathbf{A}\mathbf{x}\| \leq M\|\mathbf{x}\|$.

(b) Use the inequality derived in part (a) to show that a linear transformation $\mathbf{T} : \mathbb{R}^n \to \mathbb{R}^m$ with matrix $[T_{ij}]$ is continuous.

(c) Let $\mathbf{A} : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. If

$$\lim_{\mathbf{x}\to\mathbf{0}} \frac{\mathbf{A}\mathbf{x}}{\|\mathbf{x}\|} = \mathbf{0},$$

show that $\mathbf{A} = \mathbf{0}$.

[3]Answers, and hints for odd-numbered exercises are found at the end of this supplement, as well as selected complete solutions.

3. Let $f : A \to B$ and $g : B \to C$ be maps between open subsets of Euclidean space, and let $\mathbf{x}_0$ be in A or be a boundary point of A and $\mathbf{y}_0$ be in B or be a boundary point of B.

   (a) If $\lim_{\mathbf{x}\to\mathbf{0}} f(\mathbf{x}) = \mathbf{y}_0$ and $\lim_{\mathbf{y}\to\mathbf{y}_0} g(\mathbf{y}) = \mathbf{w}$, show that, in general, $\lim_{\mathbf{x}\to\mathbf{x}_0} g(f(\mathbf{x}))$ *need not equal* $\mathbf{w}$.

   (b) If $\mathbf{y}_0 \in B$, and $\mathbf{w} = g(\mathbf{y}_0)$, show that $\lim_{\mathbf{x}\to\mathbf{x}_0} g(f(\mathbf{x})) = \mathbf{w}$.

4. A function $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$ is said to be ***uniformly continuous*** if for every $\varepsilon > 0$ there is a $\delta > 0$ such that for all points $\mathbf{p}$ and $\mathbf{q} \in A$, the condition $\|\mathbf{p} - \mathbf{q}\| < \delta$ implies $\|f(\mathbf{p}) - f(\mathbf{q})\| < \varepsilon$. (Note that a uniformly continuous function is continuous; describe explicitly the extra property that a uniformly continuous function has.)

   (a) Prove that a linear map $\mathbf{T} : \mathbb{R}^n \to \mathbb{R}^m$ is uniformly continuous. [HINT: Use Exercise 2.]

   (b) Prove that $x \longmapsto 1/x^2$ on $(0, 1]$ is continuous, but not uniformly continuous.

5. Let $\mathbf{A} = [A_{ij}]$ be a *symmetric* $n \times n$ matrix (that is, $A_{ij} = A_{ji}$) and define $f(\mathbf{x}) = \mathbf{x} \cdot \mathbf{A}\mathbf{x}$, so $f : \mathbb{R}^n \to \mathbb{R}$. Show that $\nabla f(\mathbf{x})$ is the vector $2\mathbf{A}\mathbf{x}$.

6. The following function is graphed in Figure 2.7.2:

$$f(x, y) = \begin{cases} \dfrac{2xy^2}{\sqrt{x^2 + y^4}} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0). \end{cases}$$

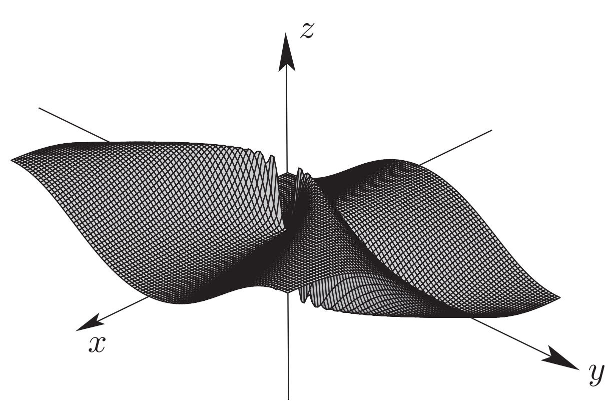

FIGURE 2.7.2. Graph of $z = 2xy^2/(x^2 + y^4)$.

Show that $\partial f/\partial x$ and $\partial f/\partial y$ exist everywhere; in fact all directional derivatives exist. But show that f is not continuous at $(0,0)$. Is f differentiable?

7. Let $f(x,y) = g(x) + h(y)$, and suppose g is differentiable at x_0 and h is differentiable at y_0. Prove from the definition that f is differentiable at (x_0, y_0).

8. Use the Cauchy-Schwarz inequality to prove the following: for any vector $\mathbf{v} \in \mathbb{R}^n$,
$$\lim_{\mathbf{x}\to\mathbf{x}_0} \mathbf{v}\cdot\mathbf{x} = \mathbf{v}\cdot\mathbf{x}_0.$$

9. Prove that if $\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = b$ for $f : A \subset \mathbb{R}^n \to \mathbb{R}$, then
$$\lim_{\mathbf{x}\to\mathbf{x}_0} [f(\mathbf{x})]^2 = b^2 \quad\text{and}\quad \lim_{\mathbf{x}\to\mathbf{x}_0} \sqrt{|f(\mathbf{x})|} = \sqrt{|b|}.$$
[You may wish to use Exercise 3(b).]

10. Show that in Theorem 9 with $m = 1$, it is enough to assume that $n - 1$ partial derivatives are continuous and merely that the other one exists. Does this agree with what you expect when $n = 1$?

11. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by
$$f(x,y) = \begin{cases} \dfrac{xy}{(x^2+y^2)^{1/2}} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0). \end{cases}$$
Show that f is continuous.

12. (a) Does $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x}{x^2+y^2}$ exist?

    (b) Does $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2}$ exist?

13. Find $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^2}{\sqrt{x^2+y^2}}$.

14. Prove that $s : \mathbb{R}^2 \to \mathbb{R}, (x,y) \longmapsto x + y$ is continuous.

15. Using the definition of continuity, prove that f is continuous at $\mathbf{x}$ if and only if
$$\lim_{\mathbf{h}\to\mathbf{0}} f(\mathbf{x}+\mathbf{h}) = f(\mathbf{x}).$$

16. (a) A sequence $\mathbf{x}_n$ of points in $\mathbb{R}^m$ is said to ***converge to*** $\mathbf{x}$, written $\mathbf{x}_n \to \mathbf{x}$ as $n \to \infty$, if for any $\varepsilon > 0$ there is an N such that $n \geq N$ implies $\|\mathbf{x} - \mathbf{x}_0\| < \varepsilon$. Show that $\mathbf{y}$ is a boundary point of an open set A if and only if $\mathbf{y}$ is not in A and there is a sequence of distinct points of A converging to $\mathbf{y}$.

(b) Let $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$ and $\mathbf{y}$ be in A or be a boundary point of A. Prove that $\lim_{\mathbf{x}\to\mathbf{y}} f(\mathbf{x}) = \mathbf{b}$ if and only if $f(\mathbf{x}_n) \to \mathbf{b}$ for every sequence $\mathbf{x}_n$ of points in A with $\mathbf{x}_n \to \mathbf{y}$.

(c) If $U \subset \mathbb{R}^m$ is open, show that $f : U \to \mathbb{R}^p$ is continuous if and only if $\mathbf{x}_n \to \mathbf{x} \in U$ implies $f(\mathbf{x}_n) \to f(\mathbf{x})$.

17. If $f(\mathbf{x}) = g(\mathbf{x})$ for all $\mathbf{x} \neq \mathbf{A}$ and if $\lim_{\mathbf{x}\to\mathbf{A}} f(\mathbf{x}) = \mathbf{b}$, then show that $\lim_{\mathbf{x}\to\mathbf{A}} g(\mathbf{x}) = \mathbf{b}$, as well.

18. Let $A \subset \mathbb{R}^n$ and let $\mathbf{x}_0$ be a boundary point of A. Let $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ be functions defined on A such that $\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x})$ and $\lim_{\mathbf{x}\to\mathbf{x}_0} g(\mathbf{x})$ exist, and assume that for all $\mathbf{x}$ in some deleted neighborhood of $\mathbf{x}_0, f(\mathbf{x}) \leq g(\mathbf{x})$. (A deleted neighborhood of $\mathbf{x}_0$ is any neighborhood of $\mathbf{x}_0$, less $\mathbf{x}_0$ itself.)

    (a) Prove that $\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) \leq \lim_{\mathbf{x}\to\mathbf{x}_0} g(\mathbf{x})$. [HINT: Consider the function $\phi(\mathbf{x}) = g(\mathbf{x}) - f(\mathbf{x})$; prove that $\lim_{\mathbf{x}\to\mathbf{x}_0} \phi(\mathbf{x}) \geq 0$, and then use the fact that the limit of the sum of two functions is the sum of their limits.]

    (b) If $f(\mathbf{x}) < g(\mathbf{x})$, do we necessarily have strict inequality of the limits?

19. Given $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$, we say that "$f$ is $o(\mathbf{x})$ as $\mathbf{x} \to \mathbf{0}$" if $\lim_{\mathbf{x}\to 0} f(\mathbf{x})/\|\mathbf{x}\| = \mathbf{0}$.

    (a) If f_1 and f_2 are $o(\mathbf{x})$ as $\mathbf{x} \to \mathbf{0}$, prove that $f_1 + f_2$ is also $o(\mathbf{x})$ as $\mathbf{x} \to \mathbf{0}$.

    (b) Let $g : A \to \mathbb{R}$ be a function with the property that there is a number $c > 0$ such that $|g(\mathbf{x}) \leq c$ for all $\mathbf{x}$ in A (the function g is said to be ***bounded***). If f is $o(\mathbf{x})$ as $\mathbf{x} \to \mathbf{0}$, prove that gf is also $o(\mathbf{x})$ as $\mathbf{x} \to \mathbf{0}$ [where $(gf)(\mathbf{x}) = g(\mathbf{x})f(\mathbf{x})$].

    (c) Show that $f(x) = x^2$ is $o(x)$ as $x \to 0$. Is $g(x) = x$ also $o(x)$ as $x \to 0$?

3
Higher-Order Derivatives and Extrema

Euler's *Analysis Infinitorum* (on analytic geometry) was followed in 1755 by the *Institutiones Calculi Differentialis*, to which it was intended as an introduction. This is the first text-book on the differential calculus which has any claim to be regarded as complete, and it may be said that until recently many modern treatises on the subject are based on it; at the same time it should be added that the exposition of the principles of the subject is often prolix and obscure, and sometimes not altogether accurate.

W. W. Rouse Ball
A Short Account of the History of Mathematics

Supplement 3.1
The Korteweg–de Vries Equation

Example. The partial differential equation $u_t + u_{xxx} + uu_x = 0$, called the *Korteweg–de Vries equation* (or KdV equation, for short), describes the motion of water waves in a shallow channel.

(a) Show that for any positive constant c, the function

$$u(x,t) = 3c\,\mathrm{sech}^2\left[\tfrac{1}{2}(x-ct)\sqrt{c}\right]$$

is a solution of the Korteweg–de Vries equation. (This solution represents a traveling "hump" of water in the channel and is called a *soliton*.)
[1]

(b) How do the shape and speed of the soliton depend on c?

♦

Solution. (a) We compute u_t, u_x, u_{xx}, and u_{xxx} using the chain rule and the differentiation formula $(d/dx)\operatorname{sech} x = -\operatorname{sech} x \tanh x$ from one-variable calculus. Letting $\alpha = (x - ct)\sqrt{c}/2$,

$$\begin{aligned} u_t &= 6c \operatorname{sech} \alpha \frac{\partial}{\partial t} \operatorname{sech} \alpha = -6c \operatorname{sech}^2 \alpha \tanh \alpha \frac{\partial \alpha}{\partial t} \\ &= 3c^{5/2} \operatorname{sech}^2 \alpha \tanh \alpha = c^{3/2} u \tanh \alpha. \end{aligned}$$

Also,

$$\begin{aligned} u_x &= -6c \operatorname{sech}^2 \alpha \tanh \alpha \frac{\partial \alpha}{\partial x} \\ &= -3c^{3/2} \operatorname{sech}^2 \alpha \tanh \alpha = -\sqrt{c}\, u \tanh \alpha, \end{aligned}$$

and so $u_t + cu_x = 0$ and

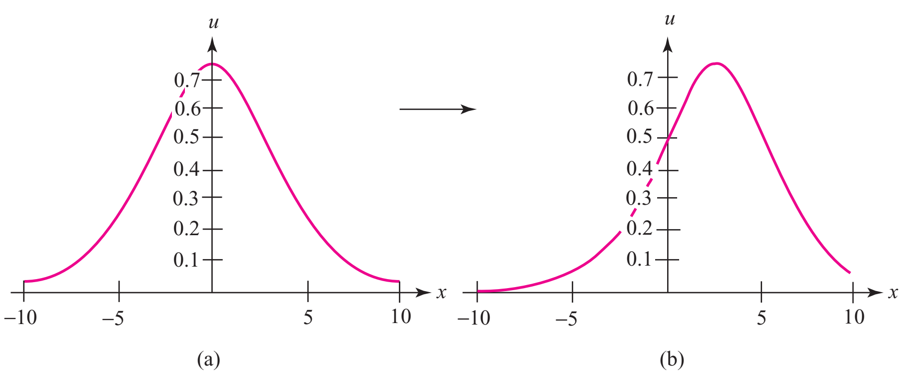

FIGURE 3.1.1. The graph of $u(x,t) = 3 \operatorname{sech}^2(\sqrt{c}(x - ct)/2)$ for $c = \frac{1}{4}$ at times (a) $t = 0$ and (b) $t = 10$.

$$\begin{aligned} u_{xx} &= -\sqrt{c}\left[u_x \tanh \alpha + u(\operatorname{sech}^2 \alpha)\frac{\sqrt{c}}{2}\right] = -\sqrt{c}\,(\tanh \alpha) u_x - \frac{u^2}{6} \\ &= c(\tanh^2 \alpha) u - \frac{u^2}{6} = c(1 - \operatorname{sech}^2 \alpha) u - \frac{u^2}{6} \\ &= cu - \frac{u^2}{3} - \frac{u^2}{6} = cu - \frac{u^2}{2}. \end{aligned}$$

[1]Solitons were first observed by J. Scott Russell around 1840 in barge canals near Edinburgh. He reported his results in *Trans. R. Soc. Edinburgh* **14** (1840): 47–109.

Thus,

$$u_{xxx} = cu_x - uu_x; \qquad \text{that is,} \qquad u_{xxx} + uu_x = cu_x.$$

Hence,

$$u_t + u_{xxx} + uu_x = u_t + cu_x = 0.$$

(b) The speed of the soliton is c, because $u(x+ct,t) = u(x,0)$. The soliton is higher and thinner when c is larger. Its shape at time $t = 10$ is shown in Figure 3.1.1. ♦

Supplement 3.3
Max Planck and The Principle of Least Action

In the 250 years after Maupertuis formulated his principle, this *principle of least action* has been found to be a "theoretical basis" for Newton's law of gravity, Maxwell's equations for electromagnetism, Schrøodinger's equation of quantum mechanics, and Einstein's field equation in general relativity.

Max Planck (see Figure 3.3.1), one of the greatest scientists of the modern era and the discoverer of the "quantization" of nature, was also a profound believer in the mathematical design of the universe. On June 29, 1922, on "Leibniz Day" in Berlin, Germany, just a few years after World War I, with all its terrible carnage, Planck delivered an address honoring this great scholar.

What follows are some excerpts from Planck's remarks:

The Theodicy culminates with the statement that whatever occurs in our world, in the large as in the small, in nature as in spiritual life, is once and for all regulated by divine reason, and in such a way that our world is the best among possible worlds. Would Leibniz reaffirm this statement even today, in view of the misery of the present time, in view of the bitter failure of many efforts not immediately aimed at material gain, in view of the undeniable fact that the imagined general harmony of people today seems to be further away from its realization than ever? No doubt, we should have to answer this question in the affirmative, even if we did not know that Leibniz never ceased to earnestly occupy himself until his last years despite an adverse fate and disappointments of all kinds, and we shall hardly err in assuming that it was exactly the *Theodicy* that gave him support and comfort in the most sorrowful days of his life. This once again is a touching example of the old truth that our most profound and most sacred principles are firmly rooted in our innermost soul, independent of experiences in the outer world.

Modern science, in particular under the influence of the development of the notion of causality, has moved far away from Leibniz's teleological point of view. Science has abandoned the assumption of a special, anticipating reason, and it considers each event in the natural and spiritual world,

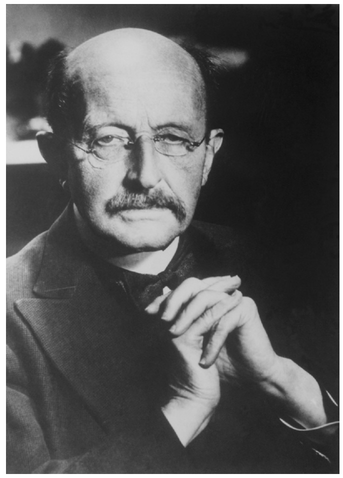

FIGURE 3.3.1. Max Planck (1858–1947).

at least in principle, as reducible to prior states. But still we notice a fact, particularly in the most exact science, which, at least in this context, is most surprising. Present-day physics, as far as it is theoretically organized, is completely governed by a system of space–time differential equations which state that each process in nature is totally determined by the events which occur in its immediate temporal and spatial neighborhood. This entire rich system of differential equations, though they differ in detail, since they refer to mechanical, electric, magnetic, and thermal processes, is now completely contained in a single dictum—*the principle of least action*. This, in short, states that, of all possible processes, the only ones that actually occur are those that involve minimum expenditure of action. As we can see, only a short step is required to recognize in the preference for the smallest quantity of action the ruling of divine reason, and thus to discover a part of Leibniz's teleological ordering of the universe.[5]

In present-day physics the principle of least action plays a relatively minor role. It does not quite fit into the framework of present theories. Of course, admittedly it is a correct statement; yet usually it serves not as the foundation of the theory, but as a true but dispensable appendix, because present theoretical physics is entirely tailored to the principle of infinitesimal local effects, and sees extensions to larger spaces and times as an unnecessary and uneconomical complication of the method of treatment. Hence, physics is inclined to view the principle of least action more as a formal and accidental curiosity than as a pillar of physical knowledge.

There is much more to the story of the least action principle, which we will revisit in Section 4.1 and in the internet supplement.[2]

[2]For more information and history, consult S. Hildebrandt and A. J. Tromba, *The Parsimonious Universe: Shape and Form in the Natural World*, Springer-Verlag, New

Supplement 3.4A
Second Derivative Test: Constrained Extrema

In this supplement, we prove Theorem 10 in §3.4. We begin by recalling the statement from the main text.

Theorem 10. *Let $f : U \subset \mathbb{R}^2 \to \mathbb{R}$ and $g : U \subset \mathbb{R}^2 \to \mathbb{R}$ be smooth (at least C^2) functions. Let $\mathbf{v}_0 \in U$, $g(\mathbf{v}_0) = c$, and let S be the level curve for g with value c. Assume that $\nabla g(\mathbf{v}_0) \neq \mathbf{0}$ and that there is a real number λ such that $\nabla f(\mathbf{v}_0) = \lambda \nabla g(\mathbf{v}_0)$. Form the auxiliary function $h = f - \lambda g$ and the bordered Hessian determinant*

$$|\bar{H}| = \begin{vmatrix} 0 & -\dfrac{\partial g}{\partial x} & -\dfrac{\partial g}{\partial y} \\[2ex] -\dfrac{\partial g}{\partial x} & \dfrac{\partial^2 h}{\partial x^2} & \dfrac{\partial^2 h}{\partial x \partial y} \\[2ex] -\dfrac{\partial g}{\partial y} & \dfrac{\partial^2 h}{\partial x \partial y} & \dfrac{\partial^2 h}{\partial y^2} \end{vmatrix} \textit{evaluated at } \mathbf{v}_0.$$

(i) *If $|\bar{H}| > 0$, then $\mathbf{v}_0$ is a local maximum point for $f|S$.*

(ii) *If $|\bar{H}| < 0$, then $\mathbf{v}_0$ is a local minimum point for $f|S$.*

(iii) *If $|\bar{H}| = 0$, the test is inconclusive and $\mathbf{v}_0$ may be a minimum, a maximum, or neither.*

The proof proceeds as follows. According to the remarks following the Lagrange multiplier theorem, the constrained extrema of f are found by looking at the critical points of the auxiliary function

$$h(x, y, \lambda) = f(x, y) - \lambda(g(x, y) - c).$$

Suppose (x_0, y_0, λ) is such a point and let $\mathbf{v}_0 = (x_0, y_0)$. That is,

$$\left.\frac{\partial f}{\partial x}\right|_{\mathbf{v}_0} = \lambda \left.\frac{\partial g}{\partial x}\right|_{\mathbf{v}_0}, \quad \left.\frac{\partial f}{\partial y}\right|_{\mathbf{v}_0} = \lambda \left.\frac{\partial g}{\partial y}\right|_{\mathbf{v}_0}, \quad \text{and} \quad g(x_0, y_0) = c.$$

In a sense this is a one-variable problem. If the function g is at all reasonable, then the set S defined by $g(x, y) = c$ is a curve and we are interested in how f varies as we move along this curve. If we can solve the equation $g(x, y) = c$ for one variable in terms of the other, then we can make this explicit and use the one-variable second-derivative test. If $\partial g/\partial y|_{\mathbf{v}_0} \neq 0$, then the curve S is not vertical at $\mathbf{v}_0$ and it is reasonable that we can solve

York/Berlin, 1995.

for y as a function of x in a neighborhood of x_0. We will, in fact, prove this in §3.5 on the implicit function theorem. (If $\partial g/\partial x|_{\mathbf{v}_0} \neq 0$, we can similarly solve for x as a function of y.)

Suppose S is the graph of $y = \phi(x)$. Then $f|S$ can be written as a function of one variable, $f(x,y) = f(x,\phi(x))$. The chain rule gives

$$\left.\begin{aligned} \frac{df}{dx} &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{d\phi}{dx} \\ \text{and} \qquad & \\ \frac{d^2 f}{dx^2} &= \frac{\partial^2 f}{\partial x^2} + 2\frac{\partial^2 f}{\partial x \partial y}\frac{d\phi}{dx} + \frac{\partial^2 f}{\partial y^2}\left(\frac{d\phi}{dx}\right)^2 + \frac{\partial f}{\partial y}\frac{d^2\phi}{dx^2}. \end{aligned}\right\} \quad (1)$$

The relation $g(x,\phi(x)) = c$ can be used to find $d\phi/dx$ and $d^2\phi/dx^2$. Differentiating both sides of $g(x,\phi(x)) = c$ with respect to x gives

$$\frac{\partial g}{\partial x} + \frac{\partial g}{\partial y}\frac{d\phi}{dx} = 0$$

and

$$\frac{\partial^2 g}{\partial x^2} + 2\frac{\partial^2 g}{\partial x \partial y}\frac{d\phi}{dx} + \frac{\partial^2 g}{\partial y^2}\left(\frac{d\phi}{dx}\right)^2 + \frac{\partial g}{\partial y}\frac{d^2\phi}{dx^2} = 0,$$

so that

$$\left.\begin{aligned} \frac{d\phi}{dx} &= -\frac{\partial g/\partial x}{\partial g/\partial y} \\ \text{and} \qquad & \\ \frac{d^2\phi}{dx^2} &= -\frac{1}{\partial g/\partial y}\left[\frac{\partial^2 g}{\partial x^2} - 2\frac{\partial^2 g}{\partial x \partial y}\frac{\partial g/\partial x}{\partial g \partial y} + \frac{\partial^2 g}{\partial y^2}\left(\frac{\partial g/\partial x}{\partial g/\partial y}\right)^2\right]. \end{aligned}\right\} \quad (2)$$

Substituting equation (7) into equation (6) gives

$$\left.\begin{aligned} \frac{df}{dx} &= \frac{\partial f}{\partial x} - \frac{\partial f/\partial y}{\partial g/\partial y}\frac{\partial g}{\partial x} \\ \text{and} \qquad & \\ \frac{d^2 f}{dx^2} &= \frac{1}{(\partial g/\partial y)^2}\left\{\left[\frac{d^2 f}{dx^2} - \frac{\partial f/\partial y}{\partial g/\partial y}\frac{\partial^2 g}{\partial x^2}\right]\left(\frac{\partial g}{\partial y}\right)^2\right. \\ &\quad -2\left[\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial f/\partial y}{\partial g/\partial y}\frac{\partial^2 g}{\partial x \partial y}\right]\frac{\partial g}{\partial x}\frac{\partial g}{\partial y} \\ &\quad \left. + \left[\frac{\partial^2 f}{\partial y^2} - \frac{\partial f/\partial y}{\partial g/\partial y}\frac{\partial^2 g}{\partial y^2}\right]\left(\frac{\partial g}{\partial x}\right)^2\right\}. \end{aligned}\right\} \quad (3)$$

At $\mathbf{v}_0$, we know that $\partial f/\partial y = \lambda \partial g/\partial y$ and $\partial f/\partial x = \lambda \partial g/\partial x$, and so equation (3) becomes

$$\left.\frac{df}{dx}\right|_{x_0} = \left.\frac{\partial f}{\partial x}\right|_{x_0} - \lambda\left.\frac{\partial g}{\partial x}\right|_{x_0} = 0$$

and

$$\left.\frac{d^2 f}{dx^2}\right|_{x_0} = \frac{1}{(\partial g/\partial y)^2}\left[\frac{\partial^2 h}{\partial x^2}\left(\frac{\partial g}{\partial y}\right)^2 - 2\frac{\partial^2 h}{\partial x \partial y}\frac{\partial g}{\partial x}\frac{\partial g}{\partial y} + \frac{\partial^2 h}{\partial y^2}\left(\frac{\partial g}{\partial x}\right)^2\right]$$

$$= -\frac{1}{(\partial g/\partial y)^2}\begin{vmatrix} 0 & -\dfrac{\partial g}{\partial x} & -\dfrac{\partial g}{\partial y} \\ -\dfrac{\partial g}{\partial x} & \dfrac{\partial^2 h}{\partial x^2} & \dfrac{\partial^2 h}{\partial x \partial y} \\ -\dfrac{\partial g}{\partial y} & \dfrac{\partial^2 h}{\partial x \partial y} & \dfrac{\partial^2 h}{\partial y^2} \end{vmatrix}$$

where the quantities are evaluated at x_0 and h is the auxiliary function introduced above. This 3×3 determinant is, as in the statement of Theorem 10, called a ***bordered Hessian***, and its sign is *opposite* that of $d^2 f/dx^2$. Therefore, if it is negative, we must be at a local minimum. If it is positive, we are at a local maximum; and if it is zero, the test is inconclusive.

Exercises.

1. Take the special case of the theorem in which $g(x, y) = y$, so that the level curve $g(x, y) = c$ with $c = 0$ is the x-axis. Does Theorem 10 reduce to a theorem you know from one-variable Calculus?

2. Show that the bordered Hessian of $f(x_1, \ldots, x_n)$ subject to the single constraint $g(x_1, \ldots, x_n) = c$ is the Hessian of the function

$$f(x_1, \ldots, x_n) - \lambda g(x_1, \ldots, x_n)$$

of the $n + 1$ variables $\lambda, x_1, \ldots, x_n$ (evaluated at the critical point). Can you use this observation to give another proof of the constrained second-derivative test using the unconstrained one? HINT: If λ_0 denotes the value of λ determined by the Lagrange multiplier theorem, consider the function

$$F(x_1, \ldots, x_n, \lambda) = f(x_1, \ldots, x_n) - \lambda g(x_1, \ldots, x_n) \pm (\lambda - \lambda_0)^2.$$

Supplement 3.4B
Proof of the Implicit Function Theorem

We begin by recalling the statement.

Theorem 11. Special Implicit Function Theorem. *Suppose that the function* $F:\mathbb{R}^{n+1}\to\mathbb{R}$ *has continuous partial derivatives. Denoting points in* $\mathbb{R}^{n+1}$ *by* $(\mathbf{x},z)$*, where* $\mathbf{x}\in\mathbb{R}^n$ *and* $z\in\mathbb{R}$*, assume that* $(\mathbf{x}_0,z_0)$ *satisfies*

$$F(\mathbf{x}_0,z_0)=0 \ \text{ and } \ \frac{\partial F}{\partial z}(\mathbf{x}_0,z_0)\neq 0.$$

Then there is a ball U *containing* $\mathbf{x}_0$ *in* $\mathbb{R}^n$ *and a neighborhood* V *of* z_0 *in* $\mathbb{R}$ *such that there is a unique function* $z=g(\mathbf{x})$ *defined for* $\mathbf{x}$ *in* U *and* z *in* V *that satisfies* $F(\mathbf{x},g(\mathbf{x}))=0$*. Moreover, if* $\mathbf{x}$ *in* U *and* z *in* V *satisfy* $F(\mathbf{x},z)=0$*, then* $z=g(\mathbf{x})$*. Finally,* $z=g(\mathbf{x})$ *is continuously differentiable, with the derivative given by*

$$\mathbf{D}g(\mathbf{x})=-\frac{1}{\dfrac{\partial F}{\partial z}(\mathbf{x},z)}\mathbf{D}_{\mathbf{x}}F(\mathbf{x},z)\Bigg|_{z=g(\mathbf{x})}$$

where $\mathbf{D}_{\mathbf{x}}F$ *denotes the derivative matrix of* F *with respect to the variable* $\mathbf{x}$*, that is,*

$$\mathbf{D}_{\mathbf{x}}F=\left[\frac{\partial F}{\partial x_1},\ldots,\frac{\partial F}{\partial x_n}\right];$$

in other words,

$$\frac{\partial g}{\partial x_i}=-\frac{\partial F/\partial x_i}{\partial F/\partial z},\quad i=1,\ldots,n. \tag{1}$$

We shall prove the case $n=2$, so that $F:\mathbb{R}^3\to\mathbb{R}$. The case for general n is done in a similar manner. We write $\mathbf{x}=(x,y)$ and $\mathbf{x}_0=(x_0,y_0)$. Since $(\partial F/\partial z)(x_0,y_0,z_0)\neq 0$, it is either positive or negative. Suppose for definiteness that it is positive. By continuity, we can find numbers $a>0$ and $b>0$ such that if $\|\mathbf{x}-\mathbf{x}_0\|<a$ and $|z-z_0|<a$, then $(\partial F/\partial z)(\mathbf{x},z)>b$. We can also assume that the other partial derivatives are bounded by a number M in this region, that is, $|(\partial F/\partial x)(\mathbf{x},z)|\leq M$ and $|(\partial F/\partial y)(\mathbf{x},z)|\leq M$, which also follows from continuity. Write $F(\mathbf{x},z)$ as follows

$$\begin{aligned}F(\mathbf{x},z)&=F(\mathbf{x},z)-F(\mathbf{x}_0,z_0)\\&=[F(\mathbf{x},z)-F(\mathbf{x}_0,z)]+[F(\mathbf{x}_0,z)-F(\mathbf{x}_0,z_0)].\end{aligned} \tag{2}$$

Consider the function

$$h(t)=F(t\mathbf{x}+(1-t)\mathbf{x}_0,z)$$

for fixed $\mathbf{x}$ and z. By the mean value theorem, there is a number θ between 0 and 1 such that

$$h(1) - h(0) = h'(\theta)(1-0) = h'(\theta),$$

that is, θ is such that

$$F(\mathbf{x}, z) - F(\mathbf{x}_0, z) = [\mathbf{D}_{\mathbf{x}} F(\theta_{\mathbf{x}} + (1-\theta)\mathbf{x}_0, z)](\mathbf{x} - \mathbf{x}_0).$$

Substitution of this formula into equation (2) along with a similar formula for the second term of that equation gives

$$F(\mathbf{x}, z) = [\mathbf{D}_{\mathbf{x}} F(\theta_{\mathbf{x}} + (1-\theta)\mathbf{x}_0, z)](\mathbf{x} - \mathbf{x}_0) + \left[\frac{\partial F}{\partial z}(\mathbf{x}_0, \phi z + (1-\phi) z_0)\right](z - z_0). \tag{3}$$

where ϕ is between 0 and 1. Let a_0 satisfy $0 < a_0 < a$ and choose $\delta > 0$ such that $\delta < a_0$ and $\delta < b a_0/2M$. If $|\mathbf{x} - \mathbf{x}_0| < \delta$ then both $|x - x_0|$ and $|y - y_0|$ are less than δ, so that the absolute value of each of the two terms in

$$[\mathbf{D}_{\mathbf{x}} F(\theta \mathbf{x} + (1-\theta)\mathbf{x}_0, z)](\mathbf{x} - \mathbf{x}_0) = \left[\frac{\partial F}{\partial x}(\theta_{\mathbf{x}} + (1-\theta)\mathbf{x}_0, z)\right](x - x_0) + \left[\frac{\partial F}{\partial y}(\theta_{\mathbf{x}} + (1-\theta)\mathbf{x}_0, z)\right](y - y_0)$$

is less than $M\delta < M(ba_0/2M) = ba_0/2$. Thus, $|\mathbf{x} - \mathbf{x}_0| < \delta$ implies

$$|[\mathbf{D}_{\mathbf{x}} F(\theta_{\mathbf{x}} + (1-\theta)\mathbf{x}_0, z)](\mathbf{x} - \mathbf{x}_0)| < ba_0.$$

Therefore, from equation (3) and the choice of b, $\|\mathbf{x} - \mathbf{x}_0\| < \delta$ implies that

$$F(\mathbf{x}, z_0 + a_0) > 0 \text{ and } F(\mathbf{x}, z_0 - a_0) < 0.$$

[The inequalities are reversed if $(\partial F/\partial z)(\mathbf{x}_0, z_0) < 0$.] Thus, by the intermediate value theorem applied to $F(\mathbf{x}, z)$ as a function of z for each $\mathbf{x}$, there is a z between $z_0 - a_0$ and $z_0 + a_0$ such that $F(\mathbf{x}, z) = 0$. This z is unique, since, by elementary calculus, a function with a positive derivative is strictly increasing and thus can have no more than one zero.

Let U be the open ball of radius δ and center $\mathbf{x}_0$ in $\mathbb{R}^2$ and let V be the open interval on $\mathbb{R}$ from $z_0 - a_0$ to $z_0 + a_0$. We have proved that if $\mathbf{x}$ is confined to U, there is a unique z in V such that $F(\mathbf{x}, z) = 0$. This defines the function $z = g(\mathbf{x}) = g(x, y)$ required by the theorem. We leave it to the reader to prove from this construction that $z = g(x, y)$ is a continuous function.

It remains to establish the continuous differentiability of $z = g(\mathbf{x})$. From equation (3), and since $F(\mathbf{x}, z) = 0$ and $z_0 = g(\mathbf{x}_0)$, we have

$$g(\mathbf{x}) - g(\mathbf{x}_0) = -\frac{[\mathbf{D}_{\mathbf{x}} F(\theta \mathbf{x} + (1-\theta)\mathbf{x}_0, z)](\mathbf{x} - \mathbf{x}_0)}{\dfrac{\partial F}{\partial z}(\mathbf{x}_0, \phi z + (1-\phi) z_0)}.$$

If we let $\mathbf{x} = (x_0 + h, y_0)$ then this equation becomes

$$\frac{g(x_0 + h, y_0) - g(x_0, y_0)}{h} = -\frac{\dfrac{\partial F}{\partial x}(\theta_{\mathbf{x}} + (1-\theta)\mathbf{x}_0, z)}{\dfrac{\partial F}{\partial z}(\mathbf{x}_0, \phi z + (1-\phi)z_0)}.$$

As $h \to 0$, it follows that $x \to x_0$ and $z \to z_0$, and so we get

$$\frac{\partial g}{\partial x}(x_0, y_0) = \lim_{h \to 0} \frac{g(x_0 + h, y_0) - g(x_0, y_0)}{h} = -\frac{\dfrac{\partial F}{\partial x}(x_0, z)}{\dfrac{\partial F}{\partial z}(x_0, z)}.$$

The formula

$$\frac{\partial g}{\partial y}(x_0, y_0) = -\frac{\dfrac{\partial F}{\partial y}(x_0, z)}{\dfrac{\partial F}{\partial z}(x_0, z)}$$

is proved in the same way. This derivation holds at any point (x, y) in U by the same argument, and so we have proved formula (1). Since the right-hand side of formula (1) is continuous, we have proved the theorem.

4
Vector Valued Functions

> I don't know what I may seem to the world, but, as to myself, I seem to have been only like a boy playing on the sea-shore, and diverting myself in now and then finding a smoother pebble or a prettier shell than ordinary, whilst the great ocean of truth lay all undiscovered before me.
>
> Isaac Newton, shortly before his death in 1727

Supplement 4.1A
Equilibria in Mechanics

Let $\mathbf{F}$ denote a force field defined on a certain domain U of $\mathbb{R}^3$. Thus, $\mathbf{F} : U \to \mathbb{R}^3$ is a given vector field. Let us agree that a particle (with mass m) is to move along a path $\mathbf{c}(t)$ in such a way that Newton's law holds; mass $\times$ acceleration = force; that is, the path $\mathbf{c}(t)$ is to satisfy the equation

$$m\mathbf{c}''(t) = \mathbf{F}(\mathbf{c}(t)). \tag{1}$$

If $\mathbf{F}$ is a potential field with potential V, that is, if $\mathbf{F} = -\nabla V$, then

$$\frac{1}{2}m\|\mathbf{c}'(t)\|^2 + V(\mathbf{c}(t)) = \text{constant}. \tag{2}$$

(The first term is called the ***kinetic energy.***) Indeed, by differentiating the left side of (2) using the chain rule

$$\begin{aligned}\frac{d}{dt}\left[\frac{1}{2}m\|\mathbf{c}'(t)\|^2 + V(\mathbf{c}(t))\right] &= m\mathbf{c}'(t)\cdot\mathbf{c}''(t) + \nabla V(\mathbf{c}(t))\cdot\mathbf{c}'(t)\\ &= [m\mathbf{c}''(t) + \nabla V(\mathbf{c}(t))]\cdot\mathbf{c}'(t) = 0.\end{aligned}$$

since $m\mathbf{c}''(t) = -\nabla V(\mathbf{c}(t))$. This proves formula (2).

Definition. *A point $\mathbf{x}_0 \in U$ is called a **position of equilibrium** if the force at that point is zero: $\mathbf{F} : (\mathbf{x}_0) = \mathbf{0}$. A point $\mathbf{x}_0$ that is a position of equilibrium is said to be **stable** if for every $\rho > 0$ and $\epsilon > 0$, we can choose numbers $\rho_0 > 0$ and $\epsilon_0 > 0$ such that a point situated anywhere at a distance less that ρ_0 from $\mathbf{x}_0$, after initially receiving kinetic energy in an amount less than ϵ_0, will forever remain a distance from $\mathbf{x}_0$ less then ρ and possess kinetic energy less then ϵ (see Figure 4.1.1).*

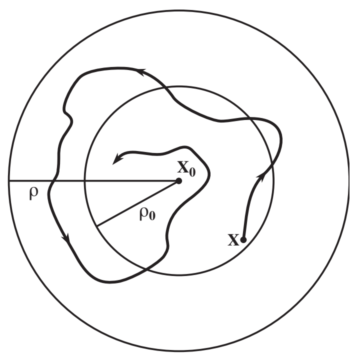

FIGURE 4.1.1. Motion near a stable point $\mathbf{x}_0$.

Thus if we have a position of equilibrium, stability at $\mathbf{x}_0$ means that a slowly moving particle near $\mathbf{x}_0$ will always remain near $\mathbf{x}_0$ and keep moving slowly. If we have an unstable equilibrium point $\mathbf{x}_0$, then $\mathbf{c}(t) = \mathbf{x}_0$ solves the equation $m\mathbf{c}''(t) = \mathbf{F}(\mathbf{c}(t))$, but nearby solutions may move away from $\mathbf{x}_0$ as time progresses. For example, a pencil balancing on its tip illustrates an unstable configuration, whereas a ball hanging on a spring illustrates a stable equilibrium.

Theorem 1.

(i) *Critical points of a potential are the positions of equilibrium.*

(ii) *In a potential field, a point $\mathbf{x}_0$ at which the potential takes a strict local minimum is a position of stable equilibrium. (Recall that a function f is said to have a* strict *local minimum at the point $\mathbf{x}_0$ if there exists* *a neighborhood U of $\mathbf{x}_0$ such that $f(\mathbf{x}) > f(\mathbf{x}_0)$ for all $\mathbf{x}$ in U other than $\mathbf{x}_0$.)*

Proof. The first assertion is quite obvious from the definition $\mathbf{F} = -\nabla V$; equilibrium points $\mathbf{x}_0$ are exactly critical points of V, at which $\nabla V(\mathbf{x}_0) = 0$.

To prove assertion (ii), we shall make use of the law of conservation of energy, that is, equation (2). We have

$$\frac{1}{2}m\|\mathbf{c}'(t)\|^2 + V(\mathbf{c}(t)) = \frac{1}{2}m\|\mathbf{c}'(t)\|^2 + V(\mathbf{c}(0)).$$

We shall argue slightly informally to amplify and illuminate the central ideas involved. Let us choose a small neighborhood of $\mathbf{x}_0$ and start our particle with a small kinetic energy. As t increases, the particle moves away from $\mathbf{x}_0$ on a path $\mathbf{c}(t)$ and $V(\mathbf{c}(t))$ increases [since $V(\mathbf{x}_0) = V(\mathbf{c}(0))$ is a strict minimum], so that the kinetic energy must decrease. If the initial kinetic energy is sufficiently small, then in order for the particle to escape from our neighborhood of $\mathbf{x}_0$, outside of which V has increased by a definite amount, the kinetic energy would have to become negative (which is impossible). Thus the particle cannot escape the neighborhood. ■

Example 1. Find the points that are positions of equilibrium, and determine whether or not they are stable, if the force field $\mathbf{F} = F_x\mathbf{i}+F_y\mathbf{j}+F_z\mathbf{k}$ is given by $F_x = -k^2x, F_y = -k^2y, F_z = -k^2z (k \neq 0)$. [1]

Solution. The force field $\mathbf{F}$ is a potential field with potential given by the function $V = \frac{1}{2}k^2(x^2 + y^2 + z^2)$. The only critical point of V is at the origin. The Hessian of V at the origin is $\frac{1}{2}k^2(h_1^2 + h_2^2 + h_3^2)$, which is positive-definite. It follows that the origin is a strict minimum of V. Thus, by (i) and (ii) of Theorem 1, we have shown that the origin is a position of stable equilibrium. ♦

Let a point in a potential field V be constrained to remain on the level surface S given by the equation $\phi(x, y, z) = 0$, with $\nabla\phi \neq \mathbf{0}$. If in formula (1) we replace $\mathbf{F}$ by the component of $\mathbf{F}$ parallel to S, we ensure that the particle will remain on S.[2] By analogy with Theorem 1, we have:

Theorem 2.

(i) *If at a point P on the surface S the potential $V|S$ has an extreme value, then the point P is a position of equilibrium on the surface.*

---

[1] The force field in this example is that governing the motion of a three-dimensional harmonic oscillator.

[2] If $\phi(x, y, z) = x^2 + y^2 + z^2 - r^2$, the particle is constrained to move on a sphere; for instance, it may be whirling on a string. The part subtracted from $\mathbf{F}$ to make it parallel to S is normal to S and is called the ***centripetal force***.

(ii) *If a point $P \in S$ is a strict local minimum of the potential $V|S$, then the point P is a position of stable equilibrium.*

The proof of this theorem will be omitted. It is similar to the proof of Theorem 1., with the additional fact that the equation of motion uses only the component of **F** along the surface. [3]

Example 2. Let **F** be the gravitational field near the surface of the earth; that is, let $\mathbf{F} = (F_x, F_y, F_z)$, where $F_x = 0, F_y = 0$, and $F_z = -mg$, where g is the acceleration due to gravity. What are the positions of equilibrium, if a particle with mass m is constrained to the sphere $\phi(x, y, z) = x^2 + y^2 + z^2 - r^2 = 0 (r > 0)$? Which of these are stable?

Solution. Notice that **F** is a potential field with $V = mgz$. Using the method of Lagrange multipliers introduced in §3.4 to locate the possible extrema, we have the equations

$$\begin{aligned} \nabla V &= \lambda \nabla \phi \\ \phi &= 0 \end{aligned}$$

or, in terms of components,

$$\begin{aligned} 0 &= 2\lambda x \\ 0 &= 2\lambda y \\ mg &= 2\lambda z \\ x^2 + y^2 + z^2 - r^2 &= 0. \end{aligned}$$

The solution of these simultaneous equations is $x = 0, y = 0, z = \pm r, \lambda = \pm mg/2r$. By Theorem 2, it follows that the points $P_1(0, 0, -r)$ and $P_2 = (0, 0, r)$ are positions of equilibrium. By observation of the potential function $V = mgz$ and by Theorem 2, part (ii), it follows that P_1 is a strict minimum and hence a stable point, whereas P_2 is not. This conclusion should be physically obvious.

♦

Exercises.

1. Let a particle move in a potential field in $\mathbb{R}^2$ given by $V(x, y) = 3x^2 + 2xy + y^2 + y + 4$. Find the stable equilibrium points, if any.

[3] These ideas can be applied to quite a number of interesting physical situations, such as molecular vibrations. The stability of such systems is an important question. For further information consult the physics literature (e.g. H. Goldstein, *Classical Mechanics*, Addison-Wesley, Reading, Mass., 1950, Chapter 10) and the mathematics literature (e.g. M. Hirsch and S. Smale, *Differential Equations, Dynamical Systems and Linear Algebra*, Academic Press, New York 1974).

2. Let a particle move in a potential field in $\mathbb{R}^2$ given by $V(x,y) = x^2 - 2xy + y^2 + y^3 + y^4$. Is the point $(0,0)$ a position of a stable equilibrium?

3. (The solution is given at the end of this Internet Supplement). Let a particle move in a potential field in $\mathbb{R}^2$ given by $V(x,y) = x^2 - 4xy - y^2 - 8x - 6y$. Find all the equilibrium points. Which, if any, are stable?

4. Let a particle be constrained to move on the circle $x^2 + y^2 = 25$ subject to the potential $V = V_1 + V_2$, where V_1 is the gravitational potential in Example 2, and $V_2(x,y) = x^2 + 24xy + 8y^2$. Find the stable equilibrium points, if any.

5. Let a particle be constrained to move on the sphere $x^2 + y^2 + z^2 = 1$, subject to the potential $V = V_1 + V_2$, where V_1 is the gravitational potential in Example 2, and $V_2(x,y) = x + y$. Find the stable equilibrium points, if any.

6. Attempt to formulate a definition and a theorem saying that if a potential has a maximum at $\mathbf{x}_0$, then $\mathbf{x}_0$ is a position of unstable equilibrium. Watch out for pitfalls in your argument.

Supplement 4.1B
Rotations and the Sunshine Formula

In this supplement we use vector methods to derive formulas for the position of the sun in the sky as a function of latitude and the day of the year.[4] To motivate this, have a look at the fascinating Figure 4.1.12 below which gives a plot of the length of day as a function of one's latitude and the day of the year. Our goal in this supplement is to use vector methods to derive this formula and to discuss related issues.

This is a nice application of vector calculus ideas because it does not involve any special technical knowledge to understand and is something that everyone can appreciate. While it is not trivial, it also does not require any particularly advanced ideas—it mainly requires patience and perseverance. Even if one does not get all the way through the details, one can learn a lot about rotations along the way.

A Bit About Rotations. Consider two unit vectors $\mathbf{l}$ and $\mathbf{r}$ in space with the same base point. If we rotate $\mathbf{r}$ about the axis passing through $\mathbf{l}$, then the tip of $\mathbf{r}$ describes a circle (Figure 4.1.2). (Imagine $\mathbf{l}$ and $\mathbf{r}$ glued rigidly at their base points and then spun about the axis through $\mathbf{l}$.) Assume that the rotation is at a uniform rate counterclockwise (when viewed from the tip of $\mathbf{l}$), making a complete revolution in T units of time. The vector $\mathbf{r}$ now is a vector function of time, so we may write $\mathbf{r} = \mathbf{c}(t)$. Our first aim is to find a convenient formula for $\mathbf{c}(t)$ in terms of its starting position $\mathbf{r}_0 = \mathbf{c}(0)$.

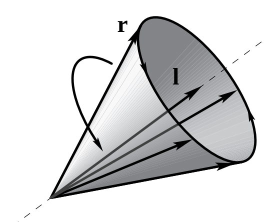

FIGURE 4.1.2. If $\mathbf{r}$ rotates about $\mathbf{l}$, its tip describes a circle.

Let λ denote the angle between $\mathbf{l}$ and $\mathbf{r}_0$; we can assume that $\lambda \neq 0$ and $\lambda \neq \pi$, *i.e.*, $\mathbf{l}$ and $\mathbf{r}_0$ are not parallel, for otherwise $\mathbf{r}$ would not rotate. In

[4] This material is adapted from *Calculus I, II, III* by J. Marsden and A. Weinstein, Springer-Verlag, New York, which can be referred to for more information and some interesting historical remarks. See also Dave Rusin's home page http://www.math.niu.edu/~rusin/ and in particular his notes on the position of the sun in the sky at http://www.math.niu.edu/~rusin/uses-math/position.sun/ We thank him for his comments on this section.

fact, we shall take λ in the open interval $(0, \pi)$. Construct the unit vector $\mathbf{m}_0$ as shown in Figure 4.1.3. From this figure we see that

$$\mathbf{r}_0 = (\cos\lambda)\mathbf{l} + (\sin\lambda)\mathbf{m}_0. \tag{1}$$

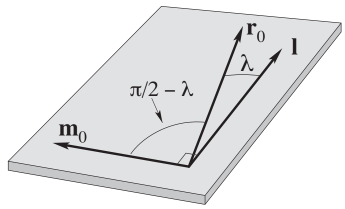

FIGURE 4.1.3. The vector $\mathbf{m}_0$ is in the plane of $\mathbf{r}_0$ and $\mathbf{l}$, is orthogonal to $\mathbf{l}$, and makes an angle of $(\pi/2) - \lambda$ with $\mathbf{r}_0$.

In fact, formula (1) can be taken as the algebraic definition of $\mathbf{m}_0$ by writing $\mathbf{m}_0 = (1/\sin\lambda)\mathbf{r}_0 - (\cos\lambda/\sin\lambda)\mathbf{l}$. We assumed that $\lambda \neq 0$, and $\lambda \neq \pi$, so $\sin\lambda \neq 0$.

Now add to this figure the unit vector $\mathbf{n}_0 = \mathbf{l} \times \mathbf{m}_0$. (See Figure 4.1.4.) The triple $(\mathbf{l}, \mathbf{m}_0, \mathbf{n}_0)$ consists of three mutually orthogonal unit vectors, just like $(\mathbf{i}, \mathbf{j}, \mathbf{k})$.

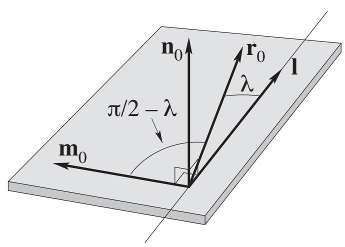

FIGURE 4.1.4. The triple $(\mathbf{l}, \mathbf{m}_0, \mathbf{n}_0)$ is a right-handed orthogonal set of unit vectors.

Example 1. Let $\mathbf{l} = (1/\sqrt{3})(\mathbf{i} + \mathbf{j} + \mathbf{k})$ and $\mathbf{r}_0 = \mathbf{k}$. Find $\mathbf{m}_0$ and $\mathbf{n}_0$.

Solution. The angle between $\mathbf{l}$ and $\mathbf{r}_0$ is given by $\cos\lambda = \mathbf{l} \cdot \mathbf{r}_0 = 1/\sqrt{3}$. This was determined by dotting both sides of formula (1) by $\mathbf{l}$ and using the fact that $\mathbf{l}$ is a unit vector. Thus, $\sin\lambda = \sqrt{1 - \cos^2\lambda} = \sqrt{2/3}$, and so from formula (1) we get

$$\begin{aligned}
\mathbf{m}_0 &= \frac{1}{\sin\lambda}\mathbf{c}(0) - \frac{\cos\lambda}{\sin\lambda}\mathbf{l} \\
&= \sqrt{\frac{3}{2}}\mathbf{k} - \frac{1}{\sqrt{3}}\sqrt{\frac{3}{2}} \cdot \frac{1}{\sqrt{3}}(\mathbf{i} + \mathbf{j} + \mathbf{k}) = \frac{2}{\sqrt{6}}\mathbf{k} - \frac{1}{\sqrt{6}}(\mathbf{i} + \mathbf{j})
\end{aligned}$$

and

$$\mathbf{n}_0 = \mathbf{l} \times \mathbf{m}_0 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & -\frac{2}{\sqrt{6}} \end{vmatrix} = \frac{1}{\sqrt{2}}\mathbf{i} - \frac{2}{\sqrt{2}}\mathbf{j}. \qquad \blacklozenge$$

Return to Figure 4.1.4 and rotate the whole picture about the axis $\mathbf{l}$. Now the "rotated" vectors $\mathbf{m}$ and $\mathbf{n}$ will vary with time as well. Since the angle λ remains constant, formula (1) applied after time t to $\mathbf{r}$ and $\mathbf{l}$ gives (see Figure 4.1.5)

$$\mathbf{m} = \frac{1}{\sin\lambda}\mathbf{r} - \frac{\cos\lambda}{\sin\lambda}\mathbf{l} \tag{2}$$

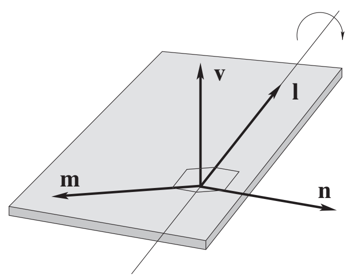

FIGURE 4.1.5. The three vectors $\mathbf{v}, \mathbf{m}$, and $\mathbf{n}$ all rotate about $\mathbf{l}$.

On the other hand, since $\mathbf{m}$ is perpendicular to $\mathbf{l}$, it rotates in a circle in the plane of $\mathbf{m}_0$ and $\mathbf{n}_0$. It goes through an angle 2π in time T, so it goes through an angle $2\pi t/T$ in t units of time, and so

$$\mathbf{m} = \cos\left(\frac{2\pi t}{T}\right)\mathbf{m}_0 + \sin\left(\frac{2\pi t}{T}\right)\mathbf{n}_0.$$

Inserting this in formula (2) and rearranging gives

$$\mathbf{r}(t) = (\cos\lambda)\mathbf{l} + \sin\lambda\cos\left(\frac{2\pi t}{T}\right)\mathbf{m}_0 + \sin\lambda\sin\left(\frac{2\pi t}{T}\right)\mathbf{n}_0. \tag{3}$$

This formula expresses explicitly how $\mathbf{r}$ changes in time as it is rotated about $\mathbf{l}$, in terms of the basic vector triple $(\mathbf{l}, \mathbf{m}_0, \mathbf{n}_0)$.

Example 2. Express the function $\mathbf{c}(t)$ explicitly in terms of $\mathbf{l}, \mathbf{r}_0$, and T.

Solution. We have $\cos\lambda = \mathbf{l} \cdot \mathbf{r}_0$ and $\sin\lambda = \|\mathbf{l} \times \mathbf{r}_0\|$. Furthermore $\mathbf{n}_0$ is a unit vector perpendicular to both $\mathbf{l}$ and $\mathbf{r}_0$, so we must have (up to an orientation)

$$\mathbf{n}_0 = \frac{\mathbf{l} \times \mathbf{r}_0}{\|\mathbf{l} \times \mathbf{r}_0\|}.$$

Thus $(\sin\lambda)\mathbf{n}_0 = \mathbf{l} \times \mathbf{r}_0$. Finally, from formula (1), we obtain $(\sin\lambda)\mathbf{m}_0 = \mathbf{r}_0 - (\cos\lambda)\mathbf{l} = \mathbf{r}_0 - (\mathbf{r}_0 \cdot \mathbf{l})\mathbf{l}$. Substituting all this into formula (3),

$$\mathbf{r} = (\mathbf{r}_0 \cdot \mathbf{l})\mathbf{l} + \cos\left(\frac{2\pi t}{T}\right)[\mathbf{r}_0 - (\mathbf{r}_0 \cdot \mathbf{l})\mathbf{l}] + \sin\left(\frac{2\pi t}{T}\right)(\mathbf{l} \times \mathbf{r}_0). \qquad \blacklozenge$$

Example 3. Show by a direct geometric argument that the speed of the tip of $\mathbf{r}$ is $(2\pi/T)\sin\lambda$. Verify that equation (3) gives the same formula.

Solution. The tip of $\mathbf{r}$ sweeps out a circle of radius $\sin\lambda$, so it covers a distance $2\pi\sin\lambda$ in time T. Its speed is therefore $(2\pi\sin\lambda)/T$ (Figure 4.1.6).

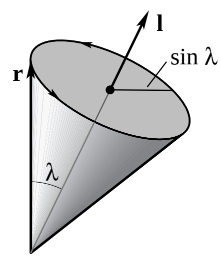

FIGURE 4.1.6. The tip of $\mathbf{r}$ sweeps out a circle of radius $\sin\lambda$.

From formula (3), we find the velocity vector to be

$$\frac{d\mathbf{r}}{dt} = -\sin\lambda \cdot \frac{2\pi}{T}\sin\left(\frac{2\pi t}{T}\right)\mathbf{m}_0 + \sin\lambda \cdot \frac{2\pi}{T}\cos\left(\frac{2\pi t}{T}\right)\mathbf{n}_0,$$

and its length is (since $\mathbf{m}_0$ and $\mathbf{n}_0$ are unit orthogonal vectors)

$$\begin{aligned}\left\|\frac{d\mathbf{r}}{dt}\right\| &= \sqrt{\sin^2\lambda \cdot \left(\frac{2\pi}{T}\right)^2 \sin^2\left(\frac{2\pi t}{T}\right) + \sin^2\lambda \cdot \left(\frac{2\pi}{T}\right)^2 \cos^2\left(\frac{2\pi t}{T}\right)} \\ &= \sin\lambda \cdot \left(\frac{2\pi}{T}\right),\end{aligned}$$

as above. $\blacklozenge$

Rotation and Revolution of the Earth. Now we apply our study of rotations to the motion of the earth about the sun, incorporating the rotation of the earth about its own axis as well. We will use a simplified model of the earth-sun system, in which the sun is fixed at the origin of our coordinate system and the earth moves at uniform speed around a circle centered at the sun. Let $\mathbf{u}$ be a unit vector pointing *from* the sun *to* the center of the earth; we have

$$\mathbf{u} = \cos(2\pi t/T_y)\mathbf{i} + \sin(2\pi t/T_y)\mathbf{j}$$

where T_y is the length of a year (t and T_y measured in the same units). See Figure 4.1.7. Notice that the unit vector pointing from the earth to the sun is $-\mathbf{u}$ and that we have oriented our axes so that $\mathbf{u} = \mathbf{i}$ when $t = 0$.

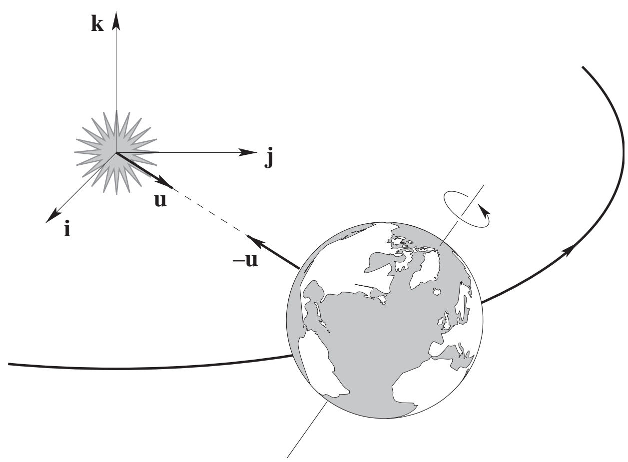

FIGURE 4.1.7. The unit vector $\mathbf{u}$ points from the sun to the earth at time t.

Next we wish to take into account the rotation of the earth. The earth rotates about an axis which we represent by a unit vector $\mathbf{l}$ pointing from the center of the earth to the North Pole. We will assume that $\mathbf{l}$ is fixed[5] with respect to $\mathbf{i}, \mathbf{j}$, and $\mathbf{k}$; astronomical measurements show that the inclination of $\mathbf{l}$ (the angle between $\mathbf{l}$ and $\mathbf{k}$) is presently about 23.5°. We will denote this angle by α. If we measure time so that the first day of summer in the northern hemisphere occurs when $t = 0$, then the axis $\mathbf{l}$ tilts in the direction $-\mathbf{i}$, and so we must have $\mathbf{l} = \cos\alpha\mathbf{k} - \sin\alpha\mathbf{i}$. (See Figure 4.1.8)

Now let $\mathbf{r}$ be the unit vector at time t from the center of the earth to a fixed point P on the earth's surface. Notice that if $\mathbf{r}$ is located with its base

[5]Actually, the axis $\mathbf{l}$ is known to rotate about $\mathbf{k}$ once every 21,000 years. This phenomenon called *precession* or *wobble*, is due to the irregular shape of the earth and may play a role in long-term climatic changes, such as ice ages. See pages. 130-134 of *The Weather Machine* by Nigel Calder, Viking (1974).

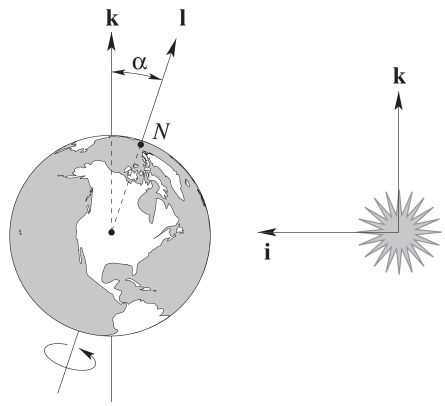

FIGURE 4.1.8. At $t = 0$, the earth's axis tilts toward the sun by the angle α.

point at P, then it represents the local *vertical* direction. We will assume that P is chosen so that at $t = 0$, it is noon at the point P; then $\mathbf{r}$ lies in the plane of $\mathbf{l}$ and $\mathbf{i}$ and makes an angle of less than 90° with $-\mathbf{i}$. Referring to Figure 4.1.9, we introduce the unit vector $\mathbf{m}_0 = -(\sin\alpha)\mathbf{k} - (\cos\alpha)\mathbf{i}$ orthogonal to $\mathbf{l}$. We then have $\mathbf{r}_0 = (\cos\lambda)\mathbf{l} + (\sin\lambda)\mathbf{m}_0$, where λ is the angle between $\mathbf{l}$ and $\mathbf{r}_0$. Since $\lambda = \pi/2 - \ell$, where ℓ is the latitude of the point P, we obtain the expression $\mathbf{r}_0 = (\sin\ell)\mathbf{l} + (\cos\ell)\mathbf{m}_0$. As in Figure 4.1.4, let $\mathbf{n}_0 = \mathbf{l} \times \mathbf{m}_0$.

Example 4. Prove that $\mathbf{n}_0 = \mathbf{l} \times \mathbf{m}_0 = -\mathbf{j}$.

Solution. Geometrically, $\mathbf{l} \times \mathbf{m}_0$ is a unit vector orthogonal to $\mathbf{l}$ and $\mathbf{m}_0$ pointing in the sense given by the right-hand rule. But $\mathbf{l}$ and $\mathbf{m}_0$ are both in the $\mathbf{i} - \mathbf{k}$ plane, so $\mathbf{l} \times \mathbf{m}_0$ points orthogonal to it in the direction $-\mathbf{j}$. (See Figure 4.1.9).

Algebraically, $\mathbf{l} = (\cos\alpha)\mathbf{k} - (\sin\alpha)\mathbf{i}$ and $\mathbf{m}_0 = -(\sin\alpha)\mathbf{k} - (\cos\alpha)\mathbf{i}$, so

$$\mathbf{l} \times \mathbf{m}_0 = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin\alpha & 0 & \cos\alpha \\ -\cos\alpha & 0 & -\sin\alpha \end{bmatrix} = -\mathbf{j}(\sin^2\alpha + \cos^2\alpha) = -\mathbf{j}. \qquad \blacklozenge$$

Now we apply formula (3) to get

$$\mathbf{r} = (\cos\lambda)\mathbf{l} + \sin\lambda\cos\left(\frac{2\pi t}{T_d}\right)\mathbf{m}_0 + \sin\lambda\sin\left(\frac{2\pi t}{T_d}\right)\mathbf{n}_0,$$

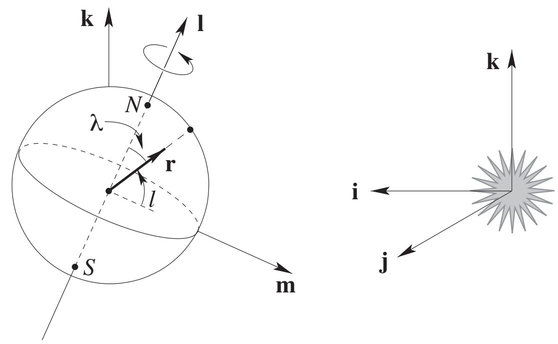

FIGURE 4.1.9. The vector **r** is the vector from the center of the earth to a fixed location P. The latitude of P is l and the colatitude is $\lambda = 90° - \ell$. The vector $\mathbf{m}_0$ is a unit vector in the plane of the equator (orthogonal to **l**) and in the plane of **l** and $\mathbf{r}_0$.

where T_d is the length of time it takes for the earth to rotate once about its axis (with respect to the "fixed stars"—*i.e.,* our $\mathbf{i}, \mathbf{j}, \mathbf{k}$ vectors). [6] Substituting the expressions derived above for $\lambda, \mathbf{l}, \mathbf{m}_0$, and $\mathbf{n}_0$, we get

$$\begin{aligned}\mathbf{r} &= \sin\ell(\cos\alpha\,\mathbf{k} - \sin\alpha\,\mathbf{i})\\ &\quad + \cos\ell\cos\left(\frac{2\pi t}{T_d}\right)(-\sin\alpha\,\mathbf{k} - \cos\alpha\,\mathbf{i}) - \cos\ell\sin\left(\frac{2\pi t}{T_d}\right)\mathbf{j}.\end{aligned}$$

Hence

$$\begin{aligned}\mathbf{r} = &- \left[\sin\ell\sin\alpha + \cos\ell\cos\alpha\cos\left(\frac{2\pi t}{T_d}\right)\right]\mathbf{i} - \cos\ell\sin\left(\frac{2\pi t}{T_d}\right)\mathbf{j}\\ &+ \left[\sin\ell\cos\alpha - \cos\ell\sin\alpha\cos\left(\frac{2\pi t}{T_d}\right)\right]\mathbf{k}. \qquad (4)\end{aligned}$$

Example 5. What is the speed (in kilometers per hour) of a point on the equator due to the rotation of the earth? A point at latitude 60 °? (The radius of the earth is 6371 kilometers.)

Solution. The speed is

$$s = (2\pi R/T_d)\sin\lambda = (2\pi R/T_d)\cos\ell,$$

where R is the radius of the earth and ℓ is the latitude. (The factor R is inserted since **r** is a *unit* vector, the actual vector from the earth's center to a point P on its surface is $R\mathbf{r}$).

[6] T_d is called the length of the *sidereal day*. It differs from the ordinary, or solar, day by about 1 part in 365 because of the rotation of the earth about the sun. In fact, $T_d \approx 23.93$ hours.

Using $T_d = 23.93$ hours and $R = 6371$ kilometers, we get $s = 1673 \cos \ell$ kilometers per hour. At the equator $\ell = 0$, so the speed is 1673 kilometers per hour; at $\ell = 60°$, $s = 836.4$ kilometers per hour. ♦

With formula (3) at our disposal, we are ready to derive the sunshine formula. The intensity of light on a portion of the earth's surface (or at the top of the atmosphere) is proportional to $\sin A$, where A is the angle of elevation of the sun above the horizon (see Fig. 4.1.10). (At night $\sin A$ is negative, and the intensity then is of course zero.)

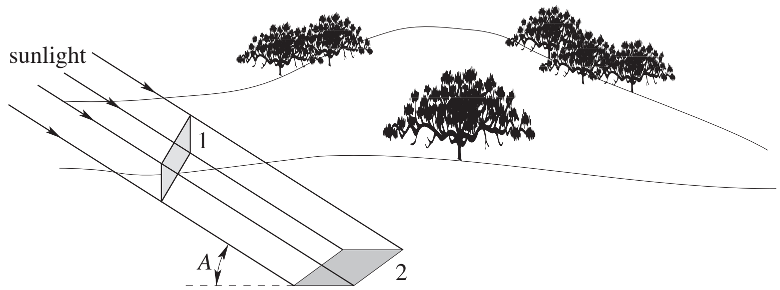

FIGURE 4.1.10. The intensity of sunlight is proportional to $\sin A$. The ratio of area 1 to area 2 is $\sin A$.

Thus we want to compute $\sin A$. From Figure 4.1.11 we see that $\sin A = -\mathbf{u} \cdot \mathbf{r}$. Substituting $\mathbf{u} = \cos(2\pi t/T_y)\mathbf{i} + \sin(2\pi t/T_y)\mathbf{j}$ and formula (3) into this formula for $\sin A$ and taking the dot product gives

$$\begin{aligned}\sin A &= \cos\left(\frac{2\pi t}{T_y}\right)\left[\sin \ell \sin \alpha + \cos \ell \cos \alpha \cos\left(\frac{2\pi t}{T_d}\right)\right] \\ &\quad + \sin\left(\frac{2\pi t}{T_y}\right)\left[\cos \ell \sin\left(\frac{2\pi t}{T_d}\right)\right] \\ &= \cos\left(\frac{2\pi t}{T_y}\right)\sin \ell \sin \alpha + \cos \ell \left[\cos\left(\frac{2\pi t}{T_y}\right)\cos \alpha \cos\left(\frac{2\pi t}{T_d}\right)\right. \\ &\quad \left. + \sin\left(\frac{2\pi t}{T_y}\right)\sin\left(\frac{2\pi t}{T_d}\right)\right]. \end{aligned} \tag{5}$$

Example 6. Set $t = 0$ in formula (4). For what ℓ is $\sin A = 0$? Interpret your result.

Solution. With $t = 0$ we get

$$\sin A = \sin \ell \sin \alpha + \cos \ell \cos \alpha = \cos(\ell - \alpha).$$

This is zero when $\ell - \alpha = \pm\pi/2$. Now $\sin A = 0$ corresponds to the sun on the horizon (sunrise or sunset), when $A = 0$ or π. Thus, at $t = 0$, this

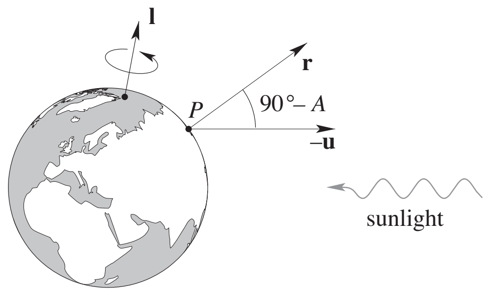

FIGURE 4.1.11. The geometry for the formula $\sin A = \cos(90° - A) = -\mathbf{u} \cdot \mathbf{r}$.

occurs when $\ell = \alpha \pm (\pi/2)$. The case $\alpha + (\pi/2)$ is impossible, since ℓ lies between $-\pi/2$ and $\pi/2$. The case $\ell = \alpha - (\pi/2)$ corresponds to a point on the Antarctic Circle; indeed at $t = 0$ (corresponding to noon on the first day of northern summer) the sun is just on the horizon at the Antarctic Circle. ♦

Our next goal is to describe the variation of $\sin A$ with time *on a particular day*. For this purpose, the time variable t is not very convenient; it will be better to measure time from noon on the day in question. To simplify our calculations, we will assume that the expressions $\cos(2\pi t/T_y)$ and $\sin(2\pi t/T_y)$ are constant over the course of any particular day; since T_y is approximately 365 times as large as the change in t, this is a reasonable approximation. On the nth day (measured from June 21), we may replace $2\pi t/T_y$ by $2\pi n/365$, and formula (4.1.4) gives

$$\sin A = (\sin \ell)P + (\cos \ell)\left[Q \cos\left(\frac{2\pi t}{T_d}\right) + R \sin\left(\frac{2\pi t}{T_d}\right)\right],$$

where

$$P = \cos(2\pi n/365)\sin\alpha, Q = \cos(2\pi n/365)\cos\alpha, \text{ and } R = \sin(2\pi n/365).$$

We will write the expression $Q\cos(2\pi t/T_d) + R\sin(2\pi t/T_d)$ in the form $U\cos[2\pi(t - t_n)/T_d]$, where t_n is the time of noon of the nth day. To find U, we use the addition formula to expand the cosine:

$$\begin{aligned} &U\cos\left[\left(\frac{2\pi t}{T_d}\right) - \left(\frac{2\pi t_n}{T_d}\right)\right] \\ &= U\left[\cos\left(\frac{2\pi t}{T_d}\right)\cos\left(\frac{2\pi t_n}{T_d}\right) + \sin\left(\frac{2\pi t}{T_d}\right)\sin\left(\frac{2\pi t_n}{T_d}\right)\right]. \end{aligned}$$

Setting this equal to $Q\cos(2\pi t/T_d) + R\sin(2\pi t/T_d)$ and comparing coefficients of $\cos 2\pi t/T_d$ and $\sin 2\pi t/T_d$ gives

$$U\cos\frac{2\pi t_n}{T_d} = Q \text{ and } U\sin\frac{2\pi t_n}{T_d} = R.$$

Squaring the two equations and adding gives[7]

$$U^2 = Q^2 + R^2 \quad \text{or} \quad U = \sqrt{Q^2 + R^2},$$

while dividing the second equation by the first gives $\tan(2\pi t_n/T_d) = R/Q$. We are interested mainly in the formula for U; substituting for Q and R gives

$$\begin{aligned} U &= \sqrt{\cos^2\left(\frac{2\pi n}{365}\right)\cos^2\alpha + \sin^2\frac{2\pi n}{365}} \\ &= \sqrt{\cos^2\left(\frac{2\pi n}{365}\right)(1-\sin^2\alpha) + \sin^2\frac{2\pi n}{365}} \\ &= \sqrt{1-\cos^2\left(\frac{2\pi n}{365}\right)\sin^2\alpha}. \end{aligned}$$

Letting τ be the time in hours from noon on the nth day so that $\tau/24 = (t - t_n)/T_d$, we substitute into formula (5) to obtain the final formula:

$$\begin{aligned} \sin A = & \sin\ell\cos\left(\frac{2\pi n}{365}\right)\sin\alpha \\ & + \cos\ell\sqrt{1-\cos^2\left(\frac{2\pi n}{365}\right)\sin^2\alpha}\cos\frac{2\pi\tau}{24}. \end{aligned} \tag{6}$$

Example 7. How high is the sun in the sky in Edinburgh (latitude 56 °) at 2 p.m. on Feb. 1?

Solution. We plug into formula (6): $\alpha = 23.5,^\circ$ $\ell = 90^\circ - 56^\circ = 34^\circ$, n = number of days after June 21 = 225, and $\tau = 2$ hours. We get $\sin A = 0.5196$, so $A = 31.3.^\circ$ ♦

Formula (4) also tells us how long days are.[8] At the time S of sunset, $A = 0$. That is,

$$\cos\left(\frac{2\pi S}{24}\right) = -\tan\ell\frac{\sin\alpha\cos(2\pi T/365)}{\sqrt{1-\sin^2\alpha\cos^2(2\pi T/365)}}. \tag{4.1.1}$$

[7] We take the positive square root because $\sin A$ should have a local maximum when $t = t_n$.

[8] If $\pi/2 - \alpha < |\ell| < \pi/2$ (inside the polar circles), there will be some values of t for which the right-hand side of formula (1) does not lie in the interval $[-1, 1]$. On the days corresponding to these values of t, the sun will never set ("midnight sun"). If $\ell = \pm\pi/2$, then $\tan\ell = \infty$, and the right-hand side does not make sense at all. This reflects the fact that, at the poles, it is either light all day or dark all day, depending upon the season.

Solving for S, and remembering that $S \geq 0$ since sunset occurs after noon, we get

$$S = \frac{12}{\pi} \cos^{-1} \left[-\tan \ell \frac{\sin \alpha \cos(2\pi T/365)}{\sqrt{1 - \sin^2 \alpha \cos^2(2\pi T/365)}} \right]. \tag{4.1.2}$$

The graph of S is shown in Figure 4.1.12.

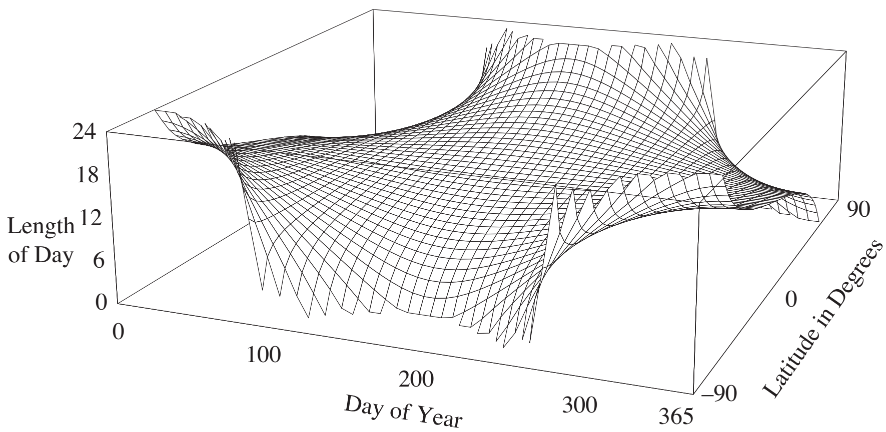

FIGURE 4.1.12. Day length as a function of latitude and day of the year.

Exercises.

1. Let $\mathbf{l} = (\mathbf{j} + \mathbf{k})/\sqrt{2}$ and $\mathbf{r}_0 = (\mathbf{i} - \mathbf{j})/\sqrt{2}$.

   (a) Find $\mathbf{m}_0$ and $\mathbf{n}_0$.

   (b) Find $\mathbf{r} = \mathbf{c}(t)$ if $T = 24$.

   (c) Find the equation of the line tangent to $\mathbf{c}(t)$ at $t = 12$ and $T = 24$.

2. From formula (3), verify that $\mathbf{c}(T/2) \cdot \mathbf{n} = 0$. Also, show this geometrically. For what values of t is $\mathbf{c}(t) \cdot \mathbf{n} = 0$?

3. If the earth rotated in the opposite direction about the sun, would T_d be longer or shorter than 24 hours? (Assume the solar day is fixed at 24 hours.)

4. Show by a direct geometric construction that

$$\mathbf{r} = \mathbf{c}(T_d/4) = -(\sin \ell \sin \alpha)\mathbf{i} - (\cos \ell)\mathbf{j} + (\sin \ell \cos \alpha)\mathbf{k}.$$

   Does this formula agree with formula (3)?

5. Derive an “exact” formula for the time of sunset from formula (4).

6. Why does formula (6) for $\sin A$ not depend on the radius of the earth? The distance of the earth from the sun?

7. How high is the sun in the sky in Paris at 3 p.m. on January 15? (The latitude of Paris is 49° N.)

8. How much solar energy (relative to a summer day at the equator) does Paris receive on January 15? (The latitude of Paris is 49° N).

9. How would your answer in Exercise 8 change if the earth were to roll to a tilt of 32° instead of 23.5°?

Supplement 4.1C
The Principle of Least Action

By Richard Feynman[9]

When I was in high school, my physics teacher—whose name was Mr. Bader—called me down one day after physics class and said, “You look bored; I want to tell you something interesting.” Then he told me something which I found absolutely fascinating, and have, since then, always found fascinating. Every time the subject comes up, I work on it. In fact, when I began to prepare this lecture I found myself making more analyses on the thing. Instead of worrying about the lecture, I got involved in a new problem. The subject is this—the principle of least action.

Mr. Bader told me the following: Suppose you have a particle (in a gravitational field, for instance) which starts somewhere and moves to some other point by free motion—you throw it, and it goes up and comes down (Figure 4.1.13).

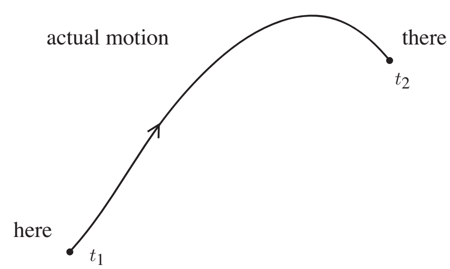

FIGURE 4.1.13.

It goes from the original place to the final place in a certain amount of time. Now, you try a different motion. Suppose that to get from here to there, it went like this (Figure 4.1.14)
but got there in just the same amount of time. Then he said this: If you calculate the kinetic energy at every moment on the path, take away the potential energy, and integrate it over the time during the whole path, you'll find that the number you'll get is bigger then that for the actual motion.

In other words, the laws of Newton could be stated not in the form $F = ma$ but in the form: the average kinetic energy less the average potential energy is as little as possible for the path of an object going from one point to another.

[9]Lecture 19 from *The Feynman Lectures on Physics.*

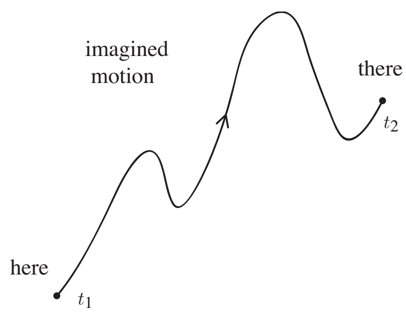

FIGURE 4.1.14.

Let me illustrate a little bit better what it means. If you take the case of the gravitational field, then if the particle has the path $x(t)$ (let's just take one dimension for a moment; we take a trajectory that goes up and down and not sideways), where x is the height above the ground, the kinetic energy is $\frac{1}{2}m(dx/dt)^2$, and the potential energy at any time is mgx. Now I take the kinetic energy minus the potential energy at every moment along the path and integrate that with respect to time from the initial time to the final time. Let's suppose that at the original time t_1 we started at some height and at the end of the time t_2 we are definitely ending at some other place (Figure 4.1.15).

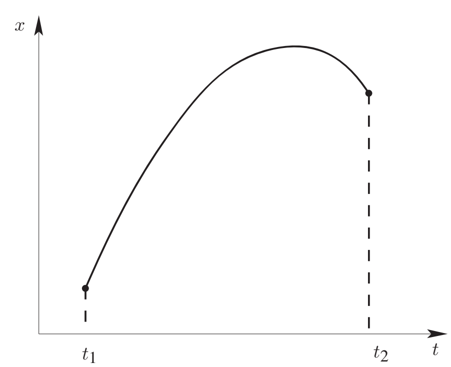

FIGURE 4.1.15.

Then the integral is

$$\int_{t_1}^{t_2} \left[\frac{1}{2}m\left(\frac{dx}{dt}\right)^2 - mgx\right] dt.$$

The actual motion is some kind of a curve—it's a parabola if we plot against the time—and gives a certain value for the integral. But we could *imagine* some other motion that went very high and came up and down in some peculiar way (Figure 4.1.16).

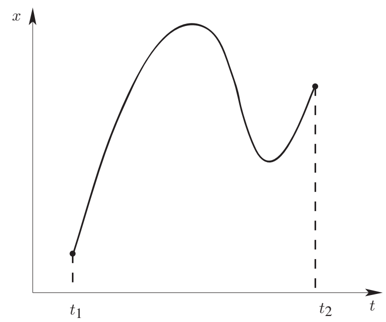

FIGURE 4.1.16.

We can calculate the kinetic energy minus the potential energy and integrate for such a path. . . or for any other path we want. The miracle is that the true path is the one for which that *integral is least.*

Let's try it out. First, suppose we take the case of a free particle for which there is no potential energy at all. Then the rule says that in going from one point to another in a given amount of time, the kinetic energy integral is least, so it must go at a uniform speed. (We know that's the right answer—to go at a uniform speed.) Why is that? Because if the particle were to go any other way, the velocities would be sometimes higher and sometimes lower then the average. The average velocity is the same for every case because it has to get from 'here' to 'there' in a given amount of time.

As and example, say your job is to start from home and get to school in a given length of time with the car. You can do it several ways: You can accelerate like mad at the beginning and slow down with the breaks near the end, or you can go at a uniform speed, or you can go backwards for a while and then go forward, and so on. The thing is that the average speed has got to be, of course, the total distance that you have gone over the time. But if you do anything but go at a uniform speed, then sometimes you are going too fast and sometimes you are going too slow. Now the mean *square* of something that deviates around an average, as you know, is always greater than the square of the mean; so the kinetic energy integral would always be higher if you wobbled your velocity than if you went at a uniform velocity. So we see that the integral is a minimum if the velocity is a constant (when there are no forces). The correct path is like this (Figure 4.1.17).

Now, an object thrown up in a gravitational field does rise faster first and then slow down. That is because there is also the potential energy, and we must have the least *difference* of kinetic and potential energy on the average. Because the potential energy rises as we go up in space, we will

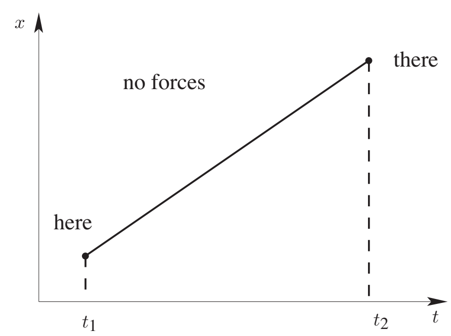

FIGURE 4.1.17.

get a lower *difference* if we can get as soon as possible up to where there is a high potential energy. Then we can take the potential away from the kinetic energy and get a lower average. So it is better to take a path which goes up and gets a lot of negative stuff from the potential energy (Figure 4.1.18).

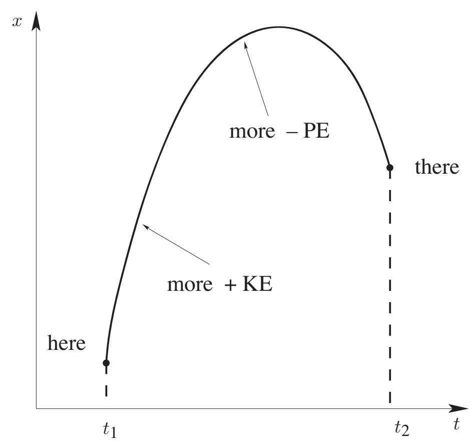

FIGURE 4.1.18.

On the other hand, you can't go up too fast, or too far, because you will then have too much kinetic energy involved—you have to go very fast to get way up and come down again in the fixed amount of time available. So you don't want to go too far up, but you want to go up some. So i turns out that the solution is some kind of balance between trying to get more potential energy with the least amount of extra kinetic energy—trying to get the difference, kinetic minus the potential, as small as possible.

That is all my teacher told me, because he was a very good teacher and knew when to stop talking. But I don't know when to stop talking. So instead of leaving it as an interesting remark, I am going to horrify and disgust you with the complexities of life by proving that it is so. The kind of mathematical problem we will have is a very difficult and a new kind. We have a certain quantity which is called the *action*, S. It is the kinetic energy, minus the potential energy, integrated over time.

$$\text{Action} = S = \int_{t_1}^{t_2} (\text{KE} - \text{PE}) dt.$$

Remember that the PE and KE are both functions of time. For each different possible path you get a different number for this action. Our mathematical problem is to find out for what curve that number is the least.

You say—Oh, that's just the ordinary calculus of maxima and minima. You calculate the action and just differentiate to find the minimum.

But watch out. Ordinarily we just have a function of some variable, and we have to find the value of what *variable* where the function is least or most. For instance, we have a rod which has been heated in the middle and the heat is spread around. For each point on the rod we have a temperature, and we must find the point at which that temperature is largest. But now for *each path in space* we have a number—quite a different thing—and we have to find the *path in space* for which the number is the minimum. That is a completely different branch of mathematics. It is not the ordinary calculus. In fact, it is called the *calculus of variations*.

There are many problems in this kind of mathematics. For example, the circle is usually defined as the locus of all points at a constant distance from a fixed point, but another way of defining a circle is this: a circle is that curve *of given length* which encloses the biggest area. Any other curve encloses less area for a given perimeter than the circle does. So if we give the problem: find that curve which encloses the greatest area for a given perimeter, we would have a problem of the calculus of variations—a different kind of calculus than you're used to.

So we make the calculation for the path of the object. Here is the way we are going to do it. The idea is that we imagine that there is a true path and that any other curve we draw is a false path, so that if we calculate the action for the false path we will get a value that is bigger that if we calculate the action for the true path (Figure 4.1.19).

Problem: Find the true path. Where is it? One way, of course, is to calculate the action for millions and millions of paths and look at which one is lowest. When you find the lowest one, that's the true path.

That's a possible way. But we can do it better than that. When we have a quantity which has a minimum—for instance, in and ordinary function like the temperature—one of the properties of the minimum is that if we go away from the minimum in the *first* order, the deviation of the function from its minimum value is only *second* order. At any place else on the curve, if we move a small distance the value of the function changes also in

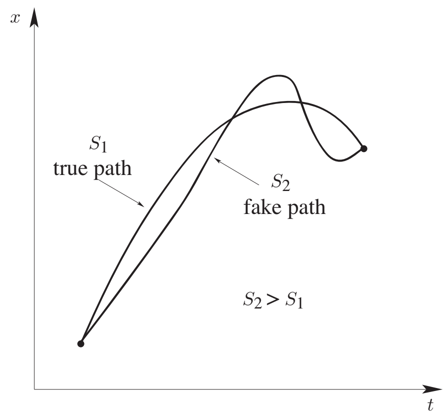

FIGURE 4.1.19.

the first order. But at a minimum, a tiny motion away makes, in the first approximation, no difference (Figure 4.1.20).

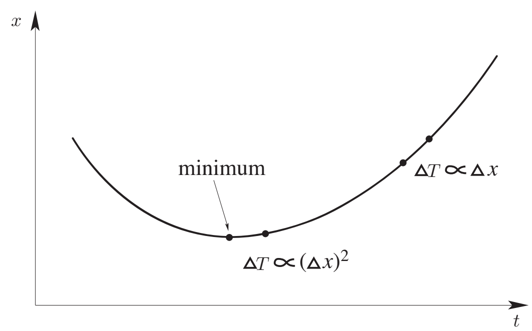

FIGURE 4.1.20.

That is what we are going to use to calculate the true path. If we have the true path, a curve which differs only a little bit from it will, in the first approximation, make no difference in the action. Any difference will be in the second approximation, if we really have a minimum.

That is easy to prove. If there is a change in the first order when I deviate the curve a certain way, there is a change in the action that is *proportional* to the deviation. The change presumably makes the action greater; otherwise we haven't got a minimum. But then if the change is *proportional* to the deviation, reversing the sign of the deviation will make the action less. We would get the action to increase one way and to decrease the other way. The only way that it could really be a minimum is that in the *first* approximation it doesn't make any change, that the changes are proportional to the square of the deviations from the true path.

So we work it this way: We call $\underline{x(t)}$ (with an underline) the true path—the one we are trying to find. We take some trial path $x(t)$ that differs from the true path by a small amount which we will call $\eta(t)$ (eta of t). (See Figure 4.1.21).

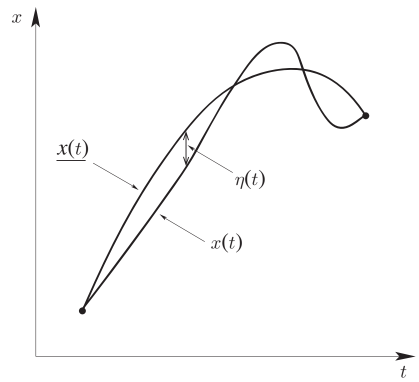

FIGURE 4.1.21.

Now the idea is that if we calculate the action S for the path $x(t)$, then the difference between that S and the action that we calculated for the path $\underline{x(t)}$—to simplify the writing we can call it $\underline{S}$—the difference of $\underline{S}$ and S must be zero in the first-order approximation of small η. It can differ in the second order, but in the first order the difference must be zero.

And that must be true for any η at all. Well, not quite. The method doesn't mean anything unless you consider paths which all begin and end at the same two points—each path begins at a certain point at t_1 and ends at a certain other point at t_2, and those points and times are kept fixed. So the deviations in our η have to be zero at each end, $\eta(t_1) = 0$ and $\eta(t_2) = 0$. With that condition, we have specified our mathematical problem.

If you didn't know any calculus, you might do the same kind of thing to find the minimum of an ordinary function $f(x)$. You could discuss what happens if you take $f(x)$ and add a small amount h to x and argue that the correction to $f(x)$ in the first order in h must be zero ant the minimum. You would substitute $x + h$ for x and expand out to the first order in h . . . just as we are going to do with η.

The idea is then that we substitute $x(t) = \underline{x(t)} + \eta(t)$ in the formula for the action:

$$S = \int \left[\frac{m}{2} \left(\frac{dx}{dt} \right)^2 - V(x) \right] dt,$$

where I call the potential energy $V(t)$. The derivative dx/dt is, of course, the derivative of $\underline{x(t)}$ plus the derivative of $\eta(t)$, so for the action I get this expression:

$$S = \int_{t_1}^{t_2} \left[\frac{m}{2} \left(\frac{d\underline{x}}{dt} + \frac{d\eta}{dt} \right)^2 - V(\underline{x} + \eta) \right] dt.$$

Now I must write this out in more detail. For the squared term I get

$$\left(\frac{d\underline{x}}{dt} \right)^2 + 2 \frac{d\underline{x}}{dt} \frac{d\eta}{dt} + \left(\frac{d\eta}{dt} \right)^2$$

But wait. I'm worrying about higher than the first order, so I will take all the terms which involve η^2 and higher powers and put them in a little box called 'second and higher order.' From this term I get only second order, but there will be more from something else. So the kinetic energy part is

$$\frac{m}{2} \left(\frac{d\underline{x}}{dt} \right)^2 + m \frac{d\underline{x}}{dt} \frac{d\eta}{dt} + (\text{second and higher order}).$$

Now we need the potential V at $\underline{x} + \eta$. I consider η small, so I can write $V(x)$ as a Taylor series. It is approximately $V(\underline{x})$; in the next approximation (from the ordinary nature of derivatives) the correction is η times the rate of change of V with respect to x, and so on:

$$V(\underline{x} + \eta) = V(\underline{x}) + \eta V'(\underline{x}) + \frac{\eta^2}{2} V''(\underline{x}) + \dots$$

I have written V' for the derivative of V with respect to x in order to save writing. The term in η^2 and the ones beyond fall into the 'second and higher order' category and we don't have to worry about them. Putting it all together,

$$S = \int_{t_1}^{t_2} \left[\frac{m}{2} \left(\frac{d\underline{x}}{dt} \right)^2 - V(\underline{x}) + m \frac{d\underline{x}}{dt} \frac{d\eta}{dt} \right.$$
$$\left. - \eta V'(\underline{x}) + (\text{second and higher order}) \right] dt.$$

Now if we look carefully at the thing, we see that the first two terms which I have arranged here correspond to the action $\underline{S}$ that I would have calculated with the true path $\underline{x}$. The thing I want to concentrate on is the change in S—the difference between the S and the $\underline{S}$ that we would get for the right path. This difference we will write as δS, called the variation in S. Leaving out the 'second and higher order' terms, I have for δS

$$\delta S = \int_{t_1}^{t_2} \left[m \frac{d\underline{x}}{dt} \frac{d\eta}{dt} - \eta V'(\underline{x}) \right] dt.$$

Now the problem is this: Here is a certain integral. I don't know what the $\underline{x}$ is yet, but I do know that *no matter what* η is, this integral must be zero. Well, you think, the only way that that can happen is that what multiplies η must be zero. But what about the first term with $d\eta/dt$? Well, after all, if η can be anything at all, its derivative is anything also, so you conclude that the coefficient of $d\eta/dt$ must also be zero. That isn't quite right. It isn't quite right because there is a connection between η and its derivative; they are not absolutely independent, because $\eta(t)$ must be zero at both t_1 and t_2.

The method of solving all problems in the calculus of variations always uses the same general principle. You make the shift in the thing you want to vary (as we did by adding η); you look at the first-order terms; *then* you always arrange things in such a form that you get an integral of the form 'some kind of stuff times the shift (η),' but with no other derivatives (no $d\eta/dt$). It must be rearranged so it is always 'something' times η. You will see the great value of that in a minute. (There are formulas that tell you how to do this in some cases without actually calculating, but they are not general enough to be worth bothering about; the best way is to calculate it out this way.)

How can I rearrange the term in $d\eta/dt$ to make it have an η? I can do that by integrating by parts. It turns out that the whole trick of the calculus of variations consists of writing down the variation of S and then integrating by parts so that the derivatives of η disappear. It is always the same in every problem in which derivatives appear.

You remember the general principle for integrating by parts. If you have any function f times $d\eta/dt$ integrated with respect to t, you write down the derivative of ηf:

$$\frac{d}{dt}(\eta f) = \eta\frac{df}{dt} + f\frac{d\eta}{dt}.$$

The integral you want is over the last term, so

$$\int f\frac{d\eta}{dt}dt = \eta f - \int \eta\frac{df}{dt}dt.$$

In our formula for δS, the function f is m times $d\underline{x}/dt$; therefore, I have the following formula for δS.

$$\delta S = m\frac{d\underline{x}}{dt}\eta(t)\bigg|_{t_1}^{t_2} - \int_{t_1}^{t_2}\frac{d}{dt}\left(m\frac{d\underline{x}}{dt}\right)\eta(t)dt - \int_{t_1}^{t_2} V'(\underline{x})\eta(t)dt.$$

The first term must be evaluated at the two limits t_1 and t_2. Then I must have the integral from the rest of the integration by parts. The last term is brought down without change.

Now comes something which always happens—the integrated part disappears. (In fact, if the integrand part does not disappear, you restate the principle, adding conditions to make sure it does!) We have already said that η must be zero at both ends of the path, because the principle is that the action is a minimum provided that the varied curve begins and ends at the chosen points. The condition is that $\eta(t_1) = 0$ and $\eta(t_2) = 0$. So the integrated term is zero. We collect the other terms together and obtain this:

$$\delta S = \int_{t_1}^{t_2} \left[-m \frac{d^2 \underline{x}}{dt^2} - V'(\underline{x}) \right] \eta(t) dt.$$

The variation in S is now the way we we wanted it—there is the stuff in brackets, say F, all multiplied by $\eta(t)$ and integrated from t_1 to t_2.

We have that an integral of something or other times $\eta(t)$ is always zero:

$$\int F(t)\eta(t) dt = 0.$$

I have some function of t; I multiply it by $\eta(t)$; and I integrate it from one end to the other. And no matter what the η is, I get zero. That means that the function $F(t)$ is zero. That's obvious, but anyway I'll show you one kind of proof.

Suppose that for $\eta(t)$ I took something which was zero for all t except right near one particular value (See Figure 4.1.22). It stays zero until it gets to this t, then it blips up for a moment and blips right back down.

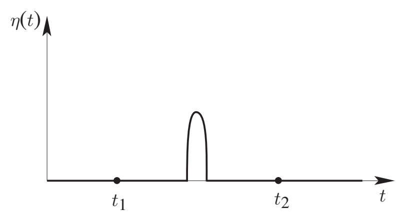

FIGURE 4.1.22.

When we do the integral of this η times any function F, the only place that you get anything other than zero was where $\eta(t)$ was blipping, and then you get the value of F at that place times the integral over the blip. The integral over the blip alone isn't zero, but when multiplied by F it has to be; so the function F has to be zero everywhere.

We see that if our integral is zero for any η, then the coefficient of η must be zero. The action integral will be a minimum for the path that satisfies this complicated differential equation:

$$\left[-m \frac{d^2 \underline{x}}{dt^2} - V'(\underline{x}) \right] = 0.$$

It's not really so complicated; you have seen it before. It is just $F = ma$. The first term is the mass times acceleration, and the second is the derivative of the potential energy, which is the force.

So, for a conservative system at least, we have demonstrated that the principle of least action gives the right answer; it says that the path that has the minimum action is the one satisfying Newton's law.

One remark: I did not prove it was a *minimum*—maybe it's a maximum. In fact, it doesn't really have to be a minimum. It is quite analogous to what we found for the 'principle of least time' which we discussed in optics. There also, we said at first it was 'least' time. It turned out, however, that there were situations in which it wasn't the *least* time. The fundamental principle was that for any *first-order variation* away from the optical path, the *change* in time was zero; it is the same story. What we really mean by 'least' is that the first-order change in the value of S, when you change the path, is zero. It is not necessarily a 'minimum'. [10]

Supplement 4.1D
Emmy Noether and Hamilton's Principle

Emmy Noether (1882–1935) (see Figure 4.1.23) is perhaps best known for her work in algebra, but she made a significant contribution to Hamilton's principle as well.[11] For planetary motion, the angular momentum vector $\mathbf{J} = \mathbf{r}(t) \times \mathrm{m}\dot{\mathbf{r}}(t)$ is time-independent (so is a *conserved quantity*), as one can readily see by computing the time derivative of $\mathbf{J}$ and using $\mathbf{F} = m\mathbf{a}$ (see Exercise 20). What Noether discovered was a deep connection between such conserved quantities and symmetries in Hamilton's principle—in the case of angular momentum, this is rotational symmetry. Noether's discoveries have had a profound influence on the study of mechanical systems, from classical to quantum, ever since.

[10] See the "*Feynman Lectures on Physics*" for the remainder of the lecture, which involves extending the above ideas to three dimensions and similar topics.

[11] "Invariante Variationsprobleme," *Göttingen Math. Phys.* **2** (1918): 235–257.

FIGURE 4.1.23. Emmy Noether (1882–1935).

Supplement to §4.3 The Orbits of Planets

The purpose of this supplement is to demonstrate that the motion of a body moving under the influence of Newton's gravitational law—that is, the inverse square force law—moves in a conic section. In the text we dealt with *circular orbits* only, but some of the results, such as Kepler's law on the relation between the period and the size of the orbit can be generalized to the case of elliptical orbits.

History. At this point it is a good idea to review some of the history given at the beginning of the text in addition to the relevant sections of the text (especially §4.1). Recall, for example, that one of the important observational points that was part of the Ptolemaic theory as well as being properly explained by the fact that planets move (to an excellent degree of approximation) in ellipses about the sun, is the appearance of retrograde motion of the planets, as in Figure 4.3.1.

Methodology. Our approach in this supplement is to use the laws of conservation of energy and conservation of angular momentum. This means that we will focus on a combination of techniques using the basic laws of mechanics together with techniques from differential equations together with a couple of tricks. Other approaches, and in fact, Newton's original approach were much more geometrical.[12]

[12]The geometric approach, along with a lot of interesting history, is also emphasized in the little book *Feynman's Lost Lecture: The Motion of Planets Around the Sun* by

FIGURE 4.3.1. the apparent motion of the planets. The photograph shows the movements of Mercury, Venus, Mars, Jupiter, and Saturn. the diagram shows the paths traced by the planets as seen from the Earth, which the Ptolemaic theory tried to explain.

The Kepler Problem. As in the text, we assume that the Sun (of mass M) is so massive that it remains fixed at the origin and our planet (with mass m) revolves about the Sun according to Newton's second law and moving in the field determined by Newton's Law of Gravitation. Therefore, the Kepler problem is the following: *Demonstrate that the solution* $\mathbf{r}(t)$ *of the equation*

$$m\ddot{\mathbf{r}} = \frac{GmM}{r^2} \tag{4.3.3}$$

is a conic section. The procedure is to suppose that we have a solution, which we will refer to as an *orbit*, and to show it is a conic section. By reversing the argument, one can also show that any conic section, with a suitable time parametrization, is also a solution.

Energy and Angular Momentum. We saw in the paragraph in §4.3 of the text entitled *Conservation of Energy and Escaping the Earth's Gravitational Field* that any solution of equation (4.3.3) obeys the law of con-

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David L. Goodstein and Judith R. Goodstein, W.W. Norton and Co., New York.

servation of energy; that is, the quantity

$$E = \frac{1}{2}m\|\dot{\mathbf{r}}\|^2 - \frac{GmM}{r} \tag{4.3.4}$$

called the ***energy*** of the solution, is constant in time.

The second fact that we will need is conservation of angular momentum, which was given in Exercise 20, §4.1. This states that the ***angular momentum*** of a solution is constant in time, namely the cross product

$$\mathbf{J} = m\mathbf{r} \times \dot{\mathbf{r}} \tag{4.3.5}$$

is time independent for each orbit.

Orbits Lie in Planes. The first thing we observe is that any solution must lie in a plane. This is simply because of conservation of angular momentum: since $\mathbf{J}$ is a constant and since $\mathbf{J} = m\mathbf{r} \times \dot{\mathbf{r}}$, it follows that $\mathbf{r}$ lies in the plane perpendicular to the (constant) vector $\mathbf{J}$.

Introducing Polar Coordinates. From the above consideration, we can assume that our orbit lies in a plane, which we can take to be the usual xy-plane. Let us introduce polar coordinates (r, θ) in that plane as usual, so that the components of the position vector $\mathbf{r}$ are given by

$$\mathbf{r} = (r\cos\theta, r\sin\theta).$$

Differentiating in t, we see that the velocity vector has components given by

$$\dot{\mathbf{r}} = (\dot{r}\cos\theta - r\dot{\theta}\sin\theta, \dot{r}\sin\theta + r\dot{\theta}\cos\theta)$$

and so one readily computes that

$$\|\dot{\mathbf{r}}\|^2 = \dot{r}^2 + r^2\dot{\theta}^2. \tag{4.3.6}$$

Substituting (4.3.6) into conservation of energy gives

$$E = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{GmM}{r} \tag{4.3.7}$$

Similarly, the angular momentum is computed by taking the cross product of $\mathbf{r}$ and $\dot{\mathbf{r}}$:

$$\mathbf{J} = m\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ r\cos\theta & r\sin\theta & 0 \\ \dot{r}\cos\theta - r\dot{\theta}\sin\theta & \dot{r}\sin\theta + r\dot{\theta}\cos\theta & 0 \end{vmatrix} \tag{4.3.8}$$

$$= mr^2\dot{\theta}\,\mathbf{k} \tag{4.3.9}$$

Thus,

$$J = mr^2\dot{\theta} \tag{4.3.10}$$

is constant in time.

Kepler's Second Law. We notice that the expression for J is closely related to the area element in polar coordinates, namely $dA = \frac{1}{2}r^2 d\theta$. In fact, measuring area from a given reference θ_0, we get

$$\frac{dA}{dt} = \frac{J}{2m} \tag{4.3.11}$$

which is a constant. Thus, we get *Kepler's second law*, namely that *for an orbit, equal areas are swept out in equal times.* Also note that this works for any central force motion law as it depends only on conservation of angular momentum.

Rewriting the Energy Equation. Next notice that the energy equation (4.3.7) can be written, with $\dot{\theta}$ eliminated using (4.3.10), as

$$E = \frac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr^2} - \frac{GmM}{r} \tag{4.3.12}$$

It will be convenient to seek a description of the orbit in polar form as $r = r(\theta)$.

A Trick. Solving differential equations to get explicit formulas often involves a combination of clever guesses, insight and luck; the situation with the Kepler problem is one of these situations. A trick that works is to introduce the new dependent variable $u = 1/r$ (of course one has to watch out for the possibility that the orbit passes through the origin, in which case r would be zero.) But watch the nice things that happen with this choice. First of all, by the chain rule,

$$\frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta}$$

and second, notice that, using the chain rule, the preceding equation and (4.3.10),

$$\begin{aligned}\dot{r} &= \frac{dr}{d\theta}\dot{\theta} = -r^2\dot{\theta}\frac{du}{d\theta} \\ &= -\frac{J}{m}\frac{du}{d\theta}\end{aligned}$$

and so from (4.3.12) we find that

$$E = \frac{J^2}{2m}\left(\frac{du}{d\theta}\right)^2 + \frac{J^2u^2}{2m} - GmMu, \tag{4.3.13}$$

that is,

$$\frac{2Em}{J^2} = \left(\frac{du}{d\theta}\right)^2 + u^2 - \frac{2GMm^2}{J^2}u, \tag{4.3.14}$$

Notice the interesting way that only u and its derivative with respect to θ appears and that the denominators containing r have been eliminated. This is the main purpose of doing the change of dependent variable from r to u.

Completing the Square. Next, we are going to eliminate the term linear in u in (4.3.14) by completing the square. Let $\alpha = GMm^2/J^2$, so that (4.3.14) becomes

$$\frac{2Em}{J^2} = \left(\frac{du}{d\theta}\right)^2 + u^2 - 2\alpha u, \tag{4.3.15}$$

Now let

$$v = u - \alpha$$

so that equation (4.3.15) becomes

$$\left(\frac{dv}{d\theta}\right)^2 + v^2 = \beta^2, \tag{4.3.16}$$

where

$$\beta^2 = \frac{2Em}{J^2} - \alpha^2.$$

We make one more change of variables to

$$w = \frac{1}{\alpha}v = \frac{1}{\alpha}u - 1$$

so that (4.3.16) becomes

$$\left(\frac{dw}{d\theta}\right)^2 + w^2 = e^2, \tag{4.3.17}$$

where

$$e^2 = \frac{\beta^2}{\alpha^2} = \frac{2Em}{\alpha^2 J^2} - 1.$$

Solving the Equation. The solution of the equation (4.3.17) is readily verified to be

$$w = e\cos(\theta - \theta_0)$$

where θ_0 is a constant of integration. In terms of the original variable r, this means that

$$r\,(e\cos(\theta - \theta_0) + 1) = \frac{1}{\alpha}. \tag{4.3.18}$$

Orbits are Conics. Equation (4.3.18) in fact is the equation of a conic written in polar coordinates, where e is the eccentricity of the orbit, which determines its shape. The quantity $l = 1/\alpha$ determines its scale and θ_0 its orientation relative to the xy-axes; it is also the angle of the closest approach to the origin.

In the case that $0 \le e < 1$ (and $E < 0$) one has an ellipse of the form

$$\frac{(x + ae)^2}{a^2} + \frac{y^2}{b^2} = 1$$

where

$$a = \frac{l}{1-e^2} = -\frac{GmM}{2E}$$

and

$$b^2 = al = -\frac{J^2}{2mE}.$$

The case $e = 0$ gives circular orbits. One also gets hyperbolic orbits if $e > 1$ and parabolic orbits in the special case when $e = 1$. Thus, as e ranges from 0 to 1, the ellipse gets a more and more elongated shape.

Kepler's 3rd Law. From Kepler's second law and the fact that the area of an ellipse is πab, one finds that the period T of an elliptical orbit is related to the semi-major axis a by

$$\left(\frac{T}{2\pi}\right)^2 = \frac{a^3}{GM}$$

which is Kepler's third law. Note that this reduces to what we saw in the text for circular orbits in the case of a circle (when a is the radius of the circle).

Supplement to §4.4
Flows and the Geometry of the Divergence

Flows of Vector Fields. It is convenient to give the unique solution through a given point at time 0 a special notation:

$$\phi(\mathbf{x}, t) = \left\{ \begin{array}{l} \text{the position of the point on the flow line} \\ \text{through the point } \mathbf{x} \text{ after time } t \text{ has elapsed.} \end{array} \right\}.$$

With $\mathbf{x}$ as the initial condition, follow along the flow line for a time period t until the new position $\phi(\mathbf{x}, t)$ is reached (see Figure 1). Alternatively,

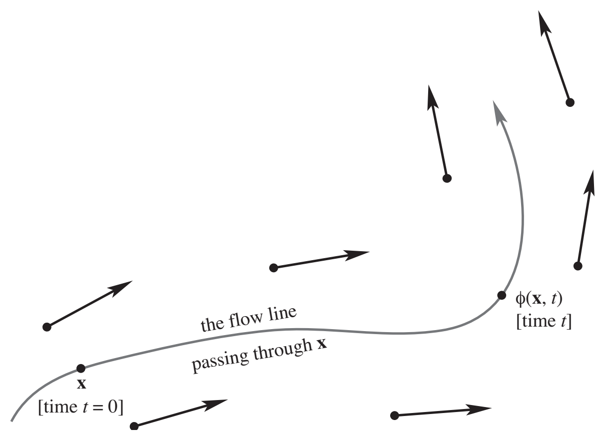


FIGURE 4.4.1. The definition of the flow $\phi(\mathbf{x}, t)$ of $\mathbf{F}$.

$\phi(\mathbf{x}, t)$ is defined by:

$$\left. \begin{array}{c} \dfrac{\partial}{\partial t}\phi(\mathbf{x}, t) = \mathbf{F}(\phi(\mathbf{x}, t)) \\ \phi(\mathbf{x}, 0) = \mathbf{x} \end{array} \right\}. \tag{1}$$

We call the mapping ϕ, which is regarded as the function of the variables $\mathbf{x}$ and t, the ***flow*** of $\mathbf{F}$.

Let $\mathbf{D_x}$ denote differentiation with respect to $\mathbf{x}$, holding t fixed. It is proved in courses on differential equations that ϕ is, in fact, a differentiable function of $\mathbf{x}$. By differentiating equation (1) with respect to $\mathbf{x}$, we get

$$\mathbf{D_x}\frac{\partial}{\partial t}\phi(\mathbf{x}, t) = \mathbf{D_x}[\mathbf{F}(\phi(\mathbf{x}, t))].$$

The equality of mixed partial derivatives may be used on the left-hand side of this equation and the chain rule applied to the right-hand side, yielding

$$\frac{\partial}{\partial t}\mathbf{D_x}\phi(\mathbf{x}, t) = \mathbf{DF}(\phi(\mathbf{x}, t))\mathbf{D_x}\phi(\mathbf{x}, t), \tag{2}$$

where $\mathbf{DF}(\phi(\mathbf{x},t))$ denotes the derivative of $\mathbf{F}$ evaluated $\phi(\mathbf{x},t)$. Equation (2), a linear differential equation for $\mathbf{D_x}\phi(\mathbf{x},t)$ is called the ***equation of the first variation***. It will be useful in our discussion of divergence and curl in the next section. Both $\mathbf{D_x F}(\phi)$ and $\mathbf{D_x}\phi$ are 3×3 matrices since $\mathbf{F}$ and ϕ take values in $\mathbb{R}^3$ and are differentiated with respect to $\mathbf{x}\in\mathbb{R}^3$; for vector fields in the plane, they would be 2×2 matrices.

The Geometry of the Divergence. We now study the geometric meaning of the divergence in more detail. This discussion depends on the concept of the flow $\phi(\mathbf{x},t)$ of a vector field $\mathbf{F}$ given in the preceding paragraph. See Exercises 3, 4, and 5 below for the corresponding discussion of the curl.

Fix a point $\mathbf{x}$ and consider the three standard basis vectors $\mathbf{i},\mathbf{j},\mathbf{k}$ emanating from $\mathbf{x}$. Let $\varepsilon > 0$ be small and consider the basis vectors $\mathbf{v}_1 = \varepsilon\mathbf{i}, \mathbf{v}_2 = \varepsilon\mathbf{j}, \mathbf{v}_3 = \varepsilon\mathbf{k}$, also emanating from $\mathbf{x}$. These vectors span a parallelepiped $P(0)$. As time increases or decreases, the flow $\phi(\mathbf{x},t)$ carries $P(0)$ into some object. For *fixed* time, ϕ is a differentiable function of $\mathbf{x}$ (that is, ϕ is a differentiable map of $\mathbb{R}^3$ to $\mathbb{R}^3$). When ε is small, the image of $P(0)$ under ϕ can be approximated by its image under the derivative of ϕ with respect to $\mathbf{x}$. Recall that if $\mathbf{v}$ is a vector based at a point P_1 and ending at P_2, so $\mathbf{v} = P_2 - P_1$, then $\phi(P_2,t) - \phi(P_1,t) \approx \mathbf{D_x}\phi(\mathbf{x},t)\cdot\mathbf{v}$. Thus for fixed time and small positive $\varepsilon, P(0)$ is approximately carried into a parallelepiped spanned by the vectors $\mathbf{v}_1(t), \mathbf{v}_2(t), \mathbf{v}_3(t)$ given by

$$\left.\begin{array}{l}\mathbf{v}_1(t) = \mathbf{D_x}\phi(\mathbf{x},t)\cdot\mathbf{v}_1\\ \mathbf{v}_2(t) = \mathbf{D_x}\phi(\mathbf{x},t)\cdot\mathbf{v}_2\\ \mathbf{v}_3(t) = \mathbf{D_x}\phi(\mathbf{x},t)\cdot\mathbf{v}_3\end{array}\right\}. \tag{3}$$

Since $\phi(\mathbf{x},0) = \mathbf{x}$ for all $\mathbf{x}$, it follows that $\mathbf{v}_1(0) = \mathbf{v}_1, \mathbf{v}_2(0) = \mathbf{v}_2$, and $\mathbf{v}_3(0) = \mathbf{v}_3$. In summary, the vectors $\mathbf{v}_1(t), \mathbf{v}_2(t), \mathbf{v}_3(t)$ span a parallelepiped $P(t)$ that moves in time (see Figure 4.4.2).

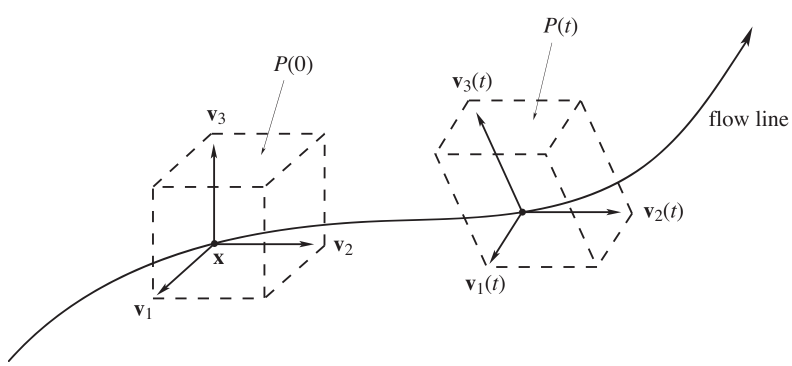


FIGURE 4.4.2. The moving basis $\mathbf{v}_1(t), \mathbf{v}_2(t), \mathbf{v}_3(t)$ and the associated parallepiped.

Let the volume of $P(t)$ be denoted by $\mathcal{V}(t)$. The main geometric meaning of divergence is given by the following theorem.

Theorem.

$$\operatorname{div} \mathbf{F}(\mathbf{x}) = \frac{1}{\mathcal{V}(0)} \frac{d}{dt}\mathcal{V}(t)\bigg|_{t=0}.$$

Proof. By equation (2) of the previous paragraph,

$$\frac{d}{dt}\mathbf{v}_i(t) = \mathbf{DF}(\phi(\mathbf{x},t)) \cdot (\mathbf{D_x}\phi(\mathbf{x},t) \cdot \mathbf{v}_i) \tag{4}$$

for $i = 1, 2, 3$. Since $\phi(\mathbf{x}, 0) = \mathbf{x}$, it follows that $\mathbf{D_x}\phi(\mathbf{x}, 0)$ is the identity matrix, so that evaluation at $t = 0$ gives

$$\frac{d}{dt}\mathbf{v}_i(t)\bigg|_{t=0} = \mathbf{DF}(\mathbf{x}) \cdot \mathbf{v}_i.$$

The volume $\mathcal{V}(t)$ is given by the triple product:

$$\mathcal{V}(t) = \mathbf{v}_1(t) \cdot [\mathbf{v}_2(t) \times \mathbf{v}_3(t)].$$

Using the differentiation rules of §4.1 and the identities

$$\mathbf{v}_1 \cdot [\mathbf{v}_2 \times \mathbf{v}_3] = \mathbf{v}_2 \cdot [\mathbf{v}_3 \times \mathbf{v}_1] = \mathbf{v}_3 \cdot [\mathbf{v}_1 \times \mathbf{v}_2],$$

equation (4) gives

$$\begin{aligned}\frac{d\mathcal{V}}{dt} &= \frac{d\mathbf{v}_1}{dt} \cdot [\mathbf{v}_2(t) \times \mathbf{v}_3(t)] + \mathbf{v}_1(t) \cdot \left[\frac{d\mathbf{v}_2}{dt} \times \mathbf{v}_3(t)\right] + \mathbf{v}_1(t) \cdot \left[\mathbf{v}_2(t) \times \frac{d\mathbf{v}_3}{dt}\right] \\ &= \frac{d\mathbf{v}_1}{dt} \cdot \left[\mathbf{v}_2(t) \times \mathbf{v}_3(t)\right] + \frac{d\mathbf{v}_2}{dt} \cdot [\mathbf{v}_3(t) \times \mathbf{v}_1(t)] + \frac{d\mathbf{v}_3}{dt} \cdot \left[\mathbf{v}_1(t) \times \mathbf{v}_2(t)\right].\end{aligned}$$

At $t = 0$, substitution from formula (3) and the facts that

$$\mathbf{v}_1 \times \mathbf{v}_2 = \varepsilon\mathbf{v}_3, \quad \mathbf{v}_3 \times \mathbf{v}_1 = \varepsilon\mathbf{v}_2 \quad \text{and} \quad \mathbf{v}_2 \times \mathbf{v}_3 = \varepsilon\mathbf{v}_1,$$

gives

$$\frac{d\mathcal{V}}{dt}\bigg|_{t=0} = \varepsilon^3[\mathbf{DF}(\mathbf{x})\mathbf{i}] \cdot \mathbf{i} + \varepsilon^3[\mathbf{DF}(\mathbf{x})\mathbf{j}] \cdot \mathbf{j} + \varepsilon^3[\mathbf{DF}(\mathbf{x})\mathbf{k}] \cdot \mathbf{k}. \tag{5}$$

Since $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$, we get $[\mathbf{DF}(\mathbf{x})\mathbf{i}] \cdot \mathbf{i} = \partial F_1/\partial x$. Similarly, the second and third terms of equation (5) are $\varepsilon^3(\partial F_2/\partial y)$ and $\varepsilon^3(\partial F_3/\partial z)$. Substituting these facts into equation (5) and dividing by $\mathcal{V}(0) = \varepsilon^3$ proves the theorem. ■

The reader who is familiar with a little more linear algebra can prove this generalization of the preceding Theorem:[13] *Let* $\mathbf{v}_1, \mathbf{v}_2$ *and* $\mathbf{v}_3$ *be any three noncoplanar (not necessarily orthonormal) vectors emanating from* $\mathbf{x}$ *that flow according to the formula*

$$\mathbf{v}_i(t) = \mathbf{D}_{\mathbf{x}}\phi(\mathbf{x}, t) \cdot \mathbf{v}_i, \quad i = 1, 2, 3.$$

The vectors $\mathbf{v}_1(t), \mathbf{v}_2(t), \mathbf{v}_3(t)$ *span a parallelepiped* $P(t)$ *with volume* $\mathcal{V}(t)$*. Then*

$$\frac{1}{\mathcal{V}(0)} \frac{d\mathcal{V}}{dt}\bigg|_{t=0} = \operatorname{div} \mathbf{F}(\mathbf{x}). \tag{6}$$

In other words, the divergence of $\mathbf{F}$ at $\mathbf{x}$ is the rate at which the volumes change, per unit volume. "Rate" refers to the rate of change with respect to time as the volumes are transported by the flow.

Exercises.

1. If $f(\mathbf{x}, t)$ is a real-valued function of $\mathbf{x}$ and t, define the ***material derivative*** of f relative to a vector field $\mathbf{F}$ as

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + \nabla f(\mathbf{x}) \cdot \mathbf{F}.$$

Show that Df/Dt is the t derivative of $f(\phi(\mathbf{x}, t), t)$ (i.e., the t derivative of f transported by the flow of $\mathbf{F}$).

2. (a) Assuming uniqueness of the flow lines through a given point at a given time, prove the following property of the flow $\phi(\mathbf{x}, t)$ of a vector field $\mathbf{F}$:

$$\phi(\mathbf{x}, t + s) = \phi(\phi(\mathbf{x}, s), t).$$

(b) What is the corresponding property for $\mathbf{D}_{\mathbf{x}}\phi$?

3. Let $\mathbf{v}$ and $\mathbf{w}$ be two vectors emanating from the origin and let them be moved by the derivative of the flow:

$$\mathbf{v}(t) = \mathbf{D}_{\mathbf{x}}\phi(\mathbf{0}, t)\mathbf{v}, \quad \mathbf{w}(t) = \mathbf{D}_{\mathbf{x}}\phi(\mathbf{0}, t)\mathbf{w},$$

so that at time $t = 0$ and at the origin $\mathbf{0}$ in $\mathbb{R}^3$,

$$\frac{d\mathbf{v}}{dt}\bigg|_{t=0} = \mathbf{D}_{\mathbf{x}}\mathbf{F}(\mathbf{0}) \cdot \mathbf{v} \quad \text{and} \quad \frac{d\mathbf{w}}{dt}\bigg|_{t=0} = \mathbf{D}_{\mathbf{x}}\mathbf{F}(\mathbf{0}) \cdot \mathbf{w}.$$

---

[13] The reader will need to know how to write the matrix of a linear transformation with respect to a given basis and be familiar with the fact that the trace of a matrix is independent of the basis.

Show that

$$\left.\frac{d}{dt}\mathbf{v}\cdot\mathbf{w}\right|_{t=0} = [\mathbf{D_x F}(\mathbf{0})\cdot\mathbf{v}]\cdot\mathbf{w} + \mathbf{v}\cdot[\mathbf{D_x F}(\mathbf{0})\cdot\mathbf{w}]$$
$$= [(\mathbf{D_x F}(\mathbf{0}) + [\mathbf{D_x F}(\mathbf{0})]^T)\mathbf{v}]\cdot\mathbf{w}].$$

4. Any matrix A can be written (uniquely) as the sum of a symmetric matrix (a matrix S is *symmetric* if $S^T = S$) and an antisymmetric matrix ($W^T = -W$) as follows:

$$A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T) = S + W.$$

In particular, for $A = \mathbf{D_x F}(\mathbf{0})$,

$$S = \frac{1}{2}[\mathbf{D_x F}(\mathbf{0}) + [\mathbf{D_x F}(\mathbf{0})]^T]$$

and

$$W = \frac{1}{2}[\mathbf{D_x F}(\mathbf{0}) + [\mathbf{D_x F}(\mathbf{0})]^T].$$

We call S the ***deformation matrix*** and W the ***rotation matrix***. Show that the entries of W are determined by

$$w_{12} = -\frac{1}{2}(\mathrm{curl}\,\mathbf{F})_3, \quad w_{23} = -\frac{1}{2}(\mathrm{curl}\,\mathbf{F})_1, \quad \text{and} \quad w_{31} = -\frac{1}{2}(\mathrm{curl}\,\mathbf{F})_2.$$

5. Let $\mathbf{w} = \frac{1}{2}(\nabla\times\mathbf{F})(\mathbf{0})$. Assume that axes are chosen so that $\mathbf{w}$ is parallel to the z axis and points in the direction of $\mathbf{k}$. Let $\mathbf{v} = \mathbf{w}\times\mathbf{r}$, where $\mathbf{r} = x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$, so that $\mathbf{v}$ is the velocity field of a rotation about the axis $\mathbf{w}$ with angular velocity $\omega = \|\mathbf{w}\|$ and with curl $\mathbf{v} = 2\mathbf{w}$. Since $\mathbf{r}$ is a function of (x, y, z), $\mathbf{v}$ is also a function of (x, y, z). Show that the derivative of $\mathbf{v}$ at the origin is given by

$$\mathbf{Dv}(\mathbf{0}) = W = \begin{bmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Interpret the result.

6. Let

$$\mathbf{V}(x, y, z) = -y\mathbf{i} + x\mathbf{j}, \quad \mathbf{W}(x, y, z) = \frac{\mathbf{V}(x, y, z)}{(x^2 + y^2)^{1/2}},$$

and

$$\mathbf{Y}(x, y, z) = \frac{\mathbf{V}(x, y, z)}{(x^2 + y^2)}.$$

(a) Compute the divergence and curl of $\mathbf{V}, \mathbf{W}$, and $\mathbf{Y}$.

(b) Find the flow lines of $\mathbf{V}, \mathbf{W}$, and $\mathbf{Y}$.

(a) How will a small paddle wheel behave in the flow of each of $\mathbf{V}, \mathbf{W}$, and $\mathbf{Y}$.

7. Let $\phi(\mathbf{x}, t)$ be the flow of a vector field $\mathbf{F}$. Let $\mathbf{x}$ and t be fixed. For small vectors $\mathbf{v}_1 = \varepsilon\mathbf{i}, \mathbf{v}_2 = \varepsilon\mathbf{j}$ and $\mathbf{v}_3 = \varepsilon\mathbf{k}$ emanating from $\mathbf{x}$, let $P(0)$ be the parallelepiped spanned by $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. Argue that for small positive $\varepsilon, P(0)$ is carried by the flow to an approximate parallelepiped spanned by $\mathbf{v}_1(t), \mathbf{v}_2(t), \mathbf{v}_3(t)$ given by formula (3).

5
Double and Triple Integrals

> Both the derivative and integral can be based on the notion of a *transition point*. For example, a function is Riemann integrable when there is a transition point between the lower and upper sums.
>
> Alan Weinstein, 1982

Supplement to §5.2
Alternative Definition of the Integral

There is another approach to the definition of the integral based on step functions that you may want to mention or have some of your better students exposed to; we present this (optional) definition first.

We say that a function $g(x, y)$ defined in $R = [a, b] \times [c, d]$ is a *step function* provided there are partitions

$$a = x_0 < x_1 < x_2 < \ldots < x_n = b$$

of the closed interval $[a, b]$ and

$$c = y_0 < y_1 < y_2 < \ldots < y_m = d$$

of the closed interval $[c, d]$ such that, in each of the mn open rectangles

$$R_{ij} = (x_{i-1}, x_i) \times (y_{j-1}, y_j),$$

the function $g(x, y)$ has a constant value k_{ij}. The graph of a generic step function is shown in Figure 5.2.1.

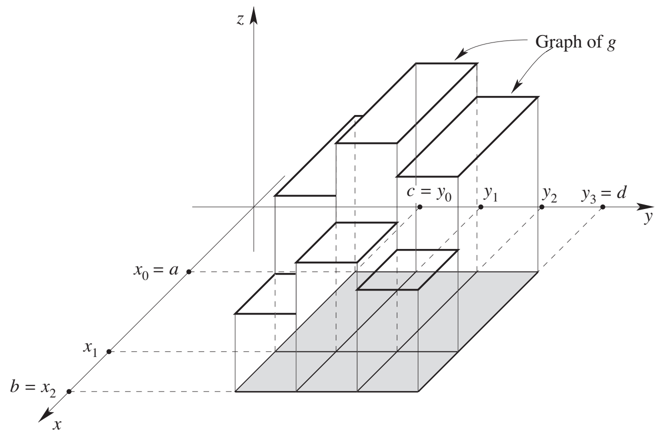


FIGURE 5.2.1. The function g is a step function since it is constant on each subrectangle.

We will define the integral

$$\iint_R g(x, y) dx\, dy$$

of a step function over a rectangle in such a way that, if $g(x, y) \geq 0$ or R, the integral equals the volume of the region V under the graph. Since the part of V lying over R_{ij} has height k_{ij} and base area $(x_i - x_{i-1})(y_i - y_{i-1})$, the volume of this part is $k_{ij}(x_i - x_{i-1})(y_i - y_{i-1})$, or $k_{ij}\Delta x_i \Delta y_j$ where $\Delta x_i = x_i - x_{i-1}$ and $\Delta y_i = y_i - y_{i-1}$ as in one-variable calculus. The volume of V is the sum of all the $k_{ij}\Delta x_i \Delta y_j$ as i ranges from 1 to n and j ranges from 1 to m (making nm terms in all). We denote this sum by

$$\sum_{i=1,j=1}^{n,m} .$$

Using this geometric guide, we *define*

$$\iint_R g(x, y) dx\, dy = \sum_{i=1,j=1}^{n,m} k_{ij}\Delta x_i \Delta y_j$$

for every step function g, whether or not its values k_{ij} are all nonnegative.

Example 1. Let g take values on the rectangles as shown in Figure 5.2.2. Calculate the integral of g over the rectangle $R = [0, 5] \times [0, 3]$.

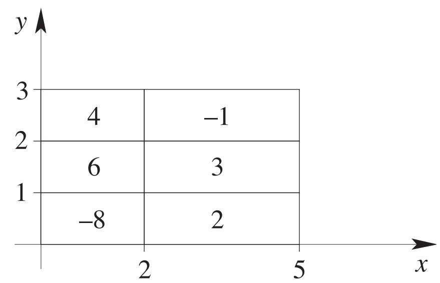


FIGURE 5.2.2. Find $\iint_D g(x,y)dx\,dy$ if g take the values shown.

Solution. The integral of g is the sum of the values of g times the areas of the rectangles:

$$\iint_R g(x,y)dx\,dy = -8\times 2 + 2\times 3 + 6\times 2 + 3\times 3 + 4\times 2 - 1\times 3 = 16.$$

♦

Of course, the functions which we want to integrate are usually not step functions. To define their integrals, we use a comparison method whose origins go back as far as Archimedes.

If $f_1(x,y)$ and $f_2(x,y)$ are any two functions defined on the rectangle R, then any reasonable definition of the double integral of f_2 should be larger than that of f_1 if $f_1(x,y) \le f_2(x,y)$ everywhere on R. Thus, if f is any function which we wish to integrate over R, and if g and h are step functions on R such that $g(x,y) \le f(x,y) \le h(x,y)$ for all (x,y) in R, then the number $\iint_R f(x,y)dx\,dy$ which we are trying to define must lie between the numbers $\iint_R g(x,y)dx\,dy$ and $\iint_R h(x,y)dx\,dy$ which have already been defined. This *condition* on the integral actually becomes a definition if we rephrase it as follows.

Alternative Definition of the Double Integral. A function f defined on a rectangle R is said to be ***integrable*** on R if for any positive number ϵ, there exist step functions $g(x,y)$ and $h(x,y)$ with $g(x,y) \le f(x,y) \le h(x,y)$ for all (x,y) in R and such that the difference

$$\iint_R h(x,y)dx\,dy - \iint_R g(x,y)dx\,dy$$

is less than ϵ. We shall call such step functions g and h "surrounding functions". When this condition holds, it can be shown that there is just one number I with the property that

$$\iint_R g(x,y)dx\,dy \le I \le \iint_R h(x,y)dx\,dy$$

for surrounding functions g and h as above. The number I is called the ***integral*** of f on R and is denoted by $\iint_R f(x,y)dx\,dy$.

Example 2. Let R be the rectangle $0 \le x \le 2, 1 \le y \le 3$, and let $f(x,y) = x^2y$. Choose a step function $h(x,y) \ge f(x,y)$ to show that $\iint_R f(x,y)dx\,dy \le 25$.

Solution. The constant function $h(x,y) = 12 \ge f(x,y)$ has integral $12 \times 4 = 48$, so we get only the crude estimate $\iint_R f(x,y)dx\,dy \le 48$. To get a better one, divide R into four pieces:

$$R_1 = [0,1] \times [1,2], \quad R_3 = [1,2] \times [1,2],$$

$$R_2 = [0,1] \times [2,3], \quad R_4 = [1,2] \times [2,3].$$

Let h be the function given by taking the maximum value of f on each subrectangle (evaluated at the upper right-hand corner); that is,

$$h(x,y) = 2 \text{ on } R_1, 3 \text{ on } R_2, 8 \text{ on } R_3, \text{ and } 12 \text{ on } R_4.$$

Therefore, the integral of h is

$$\iint_R h(x,y)dx\,dy = 2 \times 1 + 3 \times 1 + 8 \times 1 + 12 \times 1 = 25.$$

Since $h \ge f$, we get

$$\iint_R f(x,y)dx\,dy \le 25. \qquad \blacklozenge$$

Alternative Proof of Reduction to Iterated Integrals. The upper and lower sums approach can also be used to give proof of the reduction to iterated integrals. We give this (optional) proof here. We first treat step functions. Let g be a step function, with $g(x,y) = k_{ij}$ on the rectangle $(t_{i-1}, t_i) \times (s_{j-1}, s_j)$, so that

$$\iint_D g(x,y)dx\,dy = \sum_{i=1,j=1}^{n,m} k_{ij}\Delta t_i \Delta s_j.$$

If the summands $k_{ij}\Delta t_i \Delta s_j$ are laid out in a rectangular array, they may be added by first adding along rows and then adding up the subtotals, just as in the text (see Figure 5.2.3):

$$
\begin{array}{lcl}
k_{11}\Delta t_1 \Delta s_1 k_{21} \Delta t_2 \Delta s_1 \ldots k_{n1} \Delta t_n \Delta s_2 & \longrightarrow & \left(\displaystyle\sum_{i=1}^{n} k_{i1}\Delta t_i\right)\Delta s_1 \\
k_{12}\Delta t_1 \Delta s_2 k_{22} \Delta t_2 \Delta s_2 \ldots k_{n2} \Delta t_n \Delta s_2 & \longrightarrow & \left(\displaystyle\sum_{i=1}^{n} k_{i2}\Delta t_i\right)\Delta s_2 \\
\vdots \qquad \vdots \qquad \vdots & & \\
k_{1m}\Delta t_1 \Delta s_m k_{2m} \Delta t_2 \Delta s_m \ldots k_{nm} \Delta t_n \Delta s_m & \longrightarrow & \left(\displaystyle\sum_{i=1}^{n} k_{im}\Delta t_i\right)\Delta s_m \ \Big\downarrow \\
\hline
 & & \displaystyle\sum_{j=1}^{m}\left(\sum_{i=1}^{n} k_{ij}\Delta t_i\right)\Delta s_j
\end{array}
$$

FIGURE 5.2.3. Reduction to iterated integrals.

The coefficient of Δs_j in the sum over the jth row, $\displaystyle\sum_{j=1}^{n} k_{ij}\Delta t_i$, is equal to $\int_a^b g(x,y)dx$ for any y with $s_{j-1} < y < s_j$, since, for y fixed, $g(x,y)$ is a step function of x. Thus, the integral $\int_a^b g(x,y)dx$ is a step function of y, and its integral with respect to y is the sum:

$$\int_c^d \left[\int_a^b g(x,y)dx\right] dy = \sum_{j=1}^{m}\left(\sum_{i=1}^{n} k_{ij}\Delta t_i\right)\Delta s_j = \iint_D g(x,y)dx\,dy.$$

Similarly, by summing first over columns and then over rows, we obtain

$$\iint_D g(x,y)dx\,dy = \int_a^b \left[\int_c^d g(x,y)dy\right] dx.$$

The theorem is therefore true for step functions.

Now let f be integrable on $D = [a,b] \times [c,d]$ and assume that the iterated integral $\int_a^b \left[\int_c^d f(x,y)dy\right] dx$ exists. Denoting this integral by S_0, we will show that every lower sum for f on D is less than or equal to S_0, while every upper sum is greater than or equal to S_0, so S_0 must be the integral of f over D.

To carry out our program, let g be any step function such that

$$g(x,y) \le f(x,y) \tag{1}$$

for all (x, y) in D. Integrating (1) with respect to y and using "monotonicity" of the one-variable integral, we obtain

$$\int_c^d g(x,y)dy \leq \int_c^d f(x,y)dy \tag{2}$$

for all x in $[a, b]$. Integrating (2) with respect to x and applying monotonicity once more gives

$$\int_a^b \left[\int_c^d g(x,y)dy\right] dx \leq \int_a^b \left[\int_c^d f(x,y)dy\right] dx. \tag{3}$$

Since g is a step function, it follows from the first part of this proof that the left-hand side of (3) is equal to the lower sum $\iint_D g(x,y)dx\,dy$; the right-hand side of (3) is just S_0, so we have shown that every lower sum is less than or equal to S_0. The proof that every upper sum is greater than or equal to S_0 is similar, and so we are done. ■

§5.6 Technical Integration Theorems

This section provides the main ideas of the proofs of the existence and additivity of the integral that were stated in §5.2 of the text. These proofs require more advanced concepts than those needed for the rest of this chapter.

Uniform Continuity. The notions of uniform continuity and the completeness of the real numbers, both of which are usually treated more fully in a junior-level course in mathematical analysis or real-variable theory, are called upon here.

Definition. *Let* $D \subset \mathbb{R}^n$ *and* $f : D \to \mathbb{R}$. *Recall that* f *is said to be* ***continuous at*** $\mathbf{x}_0 \in D$ *provided that for all* $\varepsilon > 0$ *there is a number* $\delta > 0$ *such that if* $\mathbf{x} \in D$ *and* $\|\mathbf{x} - \mathbf{x}_0\| < \delta$, *then* $\|f(\mathbf{x}) - f(\mathbf{x}_0)\| < \varepsilon$. *We say* f *is* continuous on D *if it is continuous at each point of* D.

The function f *is said to be* ***uniformly continuous*** *on* D *if for every number* $\varepsilon > 0$ *there is a* $\delta > 0$ *such that whenever* $\mathbf{x}, \mathbf{y} \in D$ *and* $\|\mathbf{x}-\mathbf{y}\| < \delta$, *then* $\|f(\mathbf{x}) - f(\mathbf{y})\| < \varepsilon$.

The main difference between continuity and uniform continuity is that for continuity, δ can depend on $\mathbf{x}_0$ as well as ε, whereas in uniform continuity δ depends only on ε. Thus any uniformly continuous function is also continuous. An example of a function that is continuous but not uniformly continuous is given in Exercise 2 in this supplement. The distinction between the notions of continuity and uniform continuity can be rephrased: For a function f that is continuous but not uniformly continuous, δ cannot be chosen independently of the point of the domain (the $\mathbf{x}_0$ in the definition). The definition of uniform continuity states explicitly that once you are given an $\varepsilon > 0$, a δ can be found independent of any point of D.

Recall from §3.3 that a set $D \subset \mathbb{R}^n$ is ***bounded*** *if there exists a number* $M > 0$ *such that* $\|\mathbf{x}\| \leq M$ *for all* $\mathbf{x} \in D$. *A set is* **closed** *if it contains all its boundary points.* Thus a set is bounded if it can be strictly contained in some (large) ball. The next theorem states that under some conditions a continuous function is actually uniformly continuous.

Theorem. The Uniform Continuity Principle. *Every function that is continuous on a closed and bounded set* D *in* $\mathbb{R}^n$ *is uniformly continuous on* D.

The proof of this theorem will take us too far afield;[1] however, we can prove a special case of it, which is, in fact, sufficient for many situations relevant to this text.

---

[1]The proof can be found in texts on mathematical analysis. See, for example, J. Marsden and M. Hoffman, *Elementary Classical Analysis*, 2nd ed., Freeman, New York, 1993, or W. Rudin, *Principles of Mathematical Analysis*, 3rd ed., McGraw-Hill, New York, 1976.

Proof of a special case. Let us assume that $D = [a, b]$ is a closed interval on the line, that $f : D \to \mathbb{R}$ is continuous, that df/dx exists on the open interval (a, b), and that df/dx is bounded (that is, there is a constant $C > 0$ such that $|df(x)/dx| \leq C$ for all x in (a, b)). To show that these conditions imply f is uniformly continuous, we use the mean value theorem as follows: Let $\varepsilon > 0$ be given and let x and y lie on D. Then by the mean value theorem,

$$f(x) - f(y) = f'(c)(x - y)$$

for some c between x and y. By the assumed boundedness of the derivative,

$$|f(x) - f(y)| \leq C|x - y|.$$

Let $\delta = \varepsilon/C$. If $|x - y| < \delta$, then

$$|f(x) - f(y)| < C\frac{\varepsilon}{C} = \varepsilon.$$

Thus f is uniformly continuous. (Note that δ depends on neither x nor y, which is a crucial part of the definition.)

This proof also works for regions in $\mathbb{R}^n$ that are convex; that is, for any two points $\mathbf{x}, \mathbf{y}$ in D, the line segment $\mathbf{c}(t) = t\mathbf{x} + (1 - t)\mathbf{y}, 0 \leq t \leq 1$, joining them also lies in D. We assume f is differentiable (on an open set containing D) and that $\|\nabla f(\mathbf{x})\| \leq C$ for a constant C. Then the mean value theorem applied to the function $h(t) = f(\mathbf{c}(t))$ gives a point t_0 such that

$$h(1) - h(0) = [h'(t_0)][1 - 0]$$

or

$$f(\mathbf{x}) - f(\mathbf{y}) - h'(t_0) = \nabla f(\mathbf{c}(t_0)) \cdot \mathbf{c}'(t_0) = \nabla f(\mathbf{c}(t_0)) \cdot (\mathbf{x} - \mathbf{y})$$

by the chain rule. Thus by the Cauchy-Schwartz inequality,

$$|f(\mathbf{x}) - f(\mathbf{y})| \leq \|\nabla f(\mathbf{c}(t_0))\| \|\mathbf{x} - \mathbf{y}\| \leq C\|\mathbf{x} - \mathbf{y}\|.$$

Then, as above, given $\varepsilon > 0$, we can let $\delta = \varepsilon/C$. ■

Completeness. We now move on to the notion of a *Cauchy sequence* of real numbers. In the definition of Riemann sums we obtained a sequence of numbers $\{S_n\}, n = 1, \ldots$. It would be nice if we could say that this sequence of numbers converges to S (or has a limit S), but how can we obtain such a limit? In the abstract setting, we know no more about S_n than that it is the Riemann sum of a (say, continuous) function, and, although this has not yet been proved, it should be enough information to ensure its convergence.

Thus we must determine a property for sequences that guarantees their convergence. We shall define a class of sequences called Cauchy sequences, and then take as an axiom of the real number system that all such sequences converge to a limit[2]. The determination in the nineteenth century that such an axiom was necessary for the foundations of calculus was a major breakthrough in the history of mathematics and paved the way for the modern rigorous approach to mathematical analysis. We shall say more this shortly.

Definition. *A sequence of real numbers* $\{S_n\}, n = 1, \ldots,$ *is said to satisfy the* ***Cauchy criterion*** *if for every* $\varepsilon > 0$ *there exists an* N *such that for all* $m, n \geq N$*, we have* $|S_n - S_m| < \varepsilon$*.*

If a sequence S_n converges to a limit S, then S_n is a Cauchy sequence. To see this, we use the definition: For every $\varepsilon > 0$ there is an N such that for all $n \geq N, |S_n - S| < \varepsilon$. Given $\varepsilon > 0$, choose N_1 such that for $n \geq N_1, |S_n - S| < \varepsilon/2$ (use the definition with $\varepsilon/2$ in place of ε). Then if $n, m \geq N_1$,

$$|S_n - S_m| = |S_n - S + S - S_m| \leq |S_n - S| + |S - S_m| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

which proves our contention. The completeness axiom asserts that the converse is true as well:

Completeness Axiom of the Real Number. *Every Cauchy sequence* $\{S_n\}$ *converges to some limit* S*.*

Historical Note. Augustin Louis Cauchy (1789–1857), one of the greatest mathematicians of all time, defined what we now call Cauchy sequences in his *Cours d'analyse*, published in 1821. This book was a basic work on the foundations of analysis, although it would be considered somewhat loosely written by the standards of our time. Cauchy knew that a convergent sequence was "Cauchy" and remarked that a Cauchy sequence converges. He did not have a proof, nor could he have had one, since such a proof depends on the rigorous development of the real number system that was achieved only in 1872 by the German mathematician Georg Cantor (1845-1918).

It is now clear what we must do to ensure that the Riemann sums $\{S_n\}$ of, say, a continuous function on a rectangle converge to some limit S, which would prove that continuous functions on rectangles are integrable; *we must show that* $\{S_n\}$ *is a Cauchy sequence*. In demonstrating this, we use the uniform continuity principle. The integrability of continuous functions will be a consequence of the following two lemmas.

Lemma 1. *Let* f *be a continuous function on a rectangle* R *in the plane, and let* $\{S_n\}$ *be a sequence of Riemann sums for* f*. Then* $\{S_n\}$ *converges to some number* S*.*

---

[2]Texts on mathematical analysis, such as those mentioned in the preceding footnote, sometimes use different axioms, such as the least upper bound property. In such a setting, our completeness axiom becomes a theorem.

Proof. Given a rectangle $R \subset \mathbb{R}^2, R = [a,b] \times [c,d]$, we have the regular partition of $R, a = x_0 < x_1 < \cdots < x_n = b, c = y_0 < y_1 < \cdots < y_n = d$, discussed in §5.2 of the text. Recall that

$$\Delta x = x_{j+1} - x_j = \frac{b-a}{n}, \quad \Delta y = y_{k+1} - y_k = \frac{d-c}{n},$$

and

$$S_n = \sum_{j,k=0}^{n-1} f(\mathbf{c}_{jk}) \Delta x \Delta y,$$

where $\mathbf{c}_{jk}$ is an arbitrarily chosen point in $R_{jk} = [x_j, x_{j+1}] \times [y_k, y_{k+1}]$. The sequence $\{S_n\}$ is determined only by the selection of the points $\mathbf{c}_{jk}$.

For the purpose of the proof we shall introduce a slightly more complicated but very precise notation: we set

$$\Delta x^n = \frac{b-a}{n} \text{ and } \Delta y^n = \frac{d-c}{n}.$$

With this notation we have

$$S_n = \sum_{j,k}^{n-1} f(\mathbf{c}_{jk}) \Delta x^n \Delta y^n. \tag{1}$$

To show that $\{S_n\}$ satisfies the Cauchy criterion, we must show that given $\varepsilon > 0$ there exists an N such that for all $n, m \geq N, |S_n - S_m| \leq \varepsilon$. By the uniform continuity principle, f is uniformly continuous on R. Thus given $\varepsilon > 0$ there exists a $\delta > 0$ such that when $\mathbf{x}, \mathbf{y} \in R, \|\mathbf{x} - \mathbf{y}\| < \delta$, then $|f(\mathbf{x}) - f(\mathbf{y})| < \varepsilon/[2 \text{ area}(R)]$ (the quantity $\varepsilon/[2 \text{ area}(R)]$ is used in place of ε in the definition). Let N be so large that for any $m \geq N$ the diameter (length of a diagonal) of any subrectangle R_{jk} in the mth regular partition of R is less than δ. Thus if $\mathbf{x}, \mathbf{y}$ are points in the same subrectangle, we will have the inequality $|f(\mathbf{x} - f(\mathbf{y}))| < \varepsilon/[2 \text{ area}(R)]$.

Fix $m, n \geq N$. We will show that $|S_n - S_m| < \varepsilon$. This shows that $\{S_n\}$ is a Cauchy sequence and hence converges. Consider the mnth $=$ (m times n)th regular partition of R. Then

$$S_{mn} = \sum_{r,t} f(\tilde{\mathbf{c}}_{rt}) \Delta x^{mn} \Delta y^{mn},$$

where $\tilde{\mathbf{c}}_{rt}$ is a point in the rtth subrectangle. Note that each subrectangle of the mnth partition is a subrectangle of both the mth and the nth regular partitions (see Figure 5.6.1).

Let us denote the subrectangles in the mnth subdivision by $\tilde{R}_{rt}$ and those in the nth subdivision by R_{jk}. Thus each $\tilde{R}_{rt} \subset R_{jk}$ for some jk, and hence we can rewrite formula (1) as

$$S_n = \sum_{j,k}^{n-1} \left(\sum_{\tilde{R}_{rt} \subset R_{jk}} f(\mathbf{c}_{jk}) \Delta x^{mn} \Delta y^{mn} \right). \tag{1$'$}$$

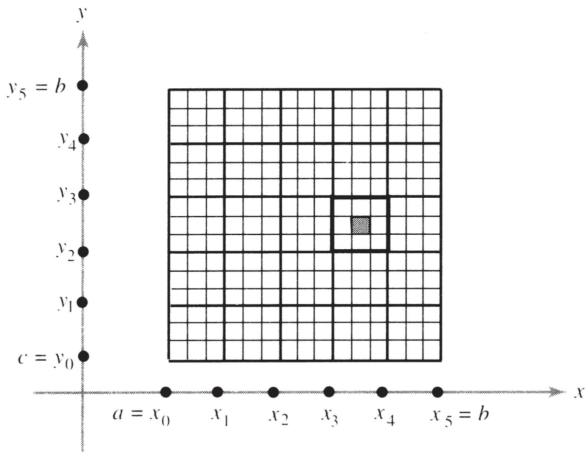


FIGURE 5.6.1. The shaded box shows a subrectangle in the mnth partition, and the darkly outlined box, a subrectangle in the mth partition.

Here we are using the fact that

$$\sum_{\tilde{R}_{rt} \subset R_{jk}} f(\mathbf{c}_{jk}) \Delta x^{mn} \Delta y^{mn} = f(\mathbf{c}_{jk}) \Delta x^n \Delta y^n,$$

where the sum is taken over all subrectangles in the mnth subdivision contained in a *fixed* rectangle R_{jk} in the nth subdivision. We also have the identity

$$S_{mn} = \sum_{r,t}^{mn-1} f(\tilde{\mathbf{c}}_{rt}) \Delta x^{mn} \Delta y^{mn}. \tag{2}$$

This relation can also be rewritten as

$$S_{mn} = \sum_{j,k} \sum_{\tilde{R}_{rt} \subset R_{jk}} f(\tilde{\mathbf{c}}_{rt}) \Delta x^{mn} \Delta y^{mn}. \tag{2'}$$

where in equation (2′) we are first summing over those subrectangles in the mnth partition contained in a fixed R_{jk} and then summing over j, k. Subtracting equation (2′) from equation (1′), we get

$$\begin{aligned} |S_n - S_{mn}| &= \left| \sum_{j,k} \sum_{\tilde{R}_{rt} \subset R_{jk}} [f(\mathbf{c}_{jk}) \Delta x^{mn} \Delta y^{mn} - f(\tilde{\mathbf{c}}_{rt}) \Delta x^{mn} \Delta y^{mn}] \right| \\ &\leq \sum_{j,k} \sum_{\tilde{R}_{rt} \subset R_{jk}} |f(\mathbf{c}_{jk}) - f(\tilde{\mathbf{c}}_{rt})| \Delta x^{mn} \Delta y^{mn}. \end{aligned}$$

By our choice of δ and N, $|f(\mathbf{c}_{jk}) - f(\tilde{\mathbf{c}}_{rt})| < \varepsilon/[2\ \text{area}(R)]$, and consequently the above inequality becomes

$$|S_n - S_{mn}| \le \sum_{j,k} \sum_{\tilde{R}_{rt} \subset R_{jk}} \frac{\varepsilon}{2\,\text{area}\,(R)} \Delta x^{mn} \Delta y^{mn} = \frac{\varepsilon}{2}.$$

Thus, $|S_n - S_{mn}| < \varepsilon/2$ and similarly one shows that $|S_n - S_{mn}| < \varepsilon/2$. Since

$$|S_n - S_m| = |S_n - S_{mn} + S_{mn} - S_m| \le |S_n - S_{mn}| + |S_{mn} - S_m| < \varepsilon$$

for $m, n \ge N$, we have shown $\{S_n\}$ satisfies the Cauchy criterion and thus has a limit S. ■

We have already remarked that each Riemann sum depends on the selection of a collection of points $\mathbf{c}_{jk}$. In order to show that a continuous function on a rectangle R is integrable, we must further demonstrate that the limit S we obtained in Lemma 1 is independent of the choices of the points $\mathbf{c}_{jk}$.

Lemma 2. *The limit S in Lemma* 1 *does not depend on the choice of points* $\mathbf{c}_{jk}$.

Proof. Suppose we have two sequences of Riemann sums $\{S_n\}$ and $\{S_n^*\}$ obtained by selecting two different sets of points, say $\mathbf{c}_{jk}$ and $\mathbf{c}_{jk}^*$ in each nth partition. By Lemma 1 we know that $\{S_n\}$ converges to some number S and $\{S_n^*\}$ must also converge to some number, say S^*. We want to show that $S = S^*$ and shall do this by showing that given any $\varepsilon > 0, |S - S^*| < \varepsilon$, which implies that S must be equal to S^* (why?).

To start, we know that f is uniformly continuous on R. Consequently, given $\varepsilon > 0$, there exits a δ such that $|f(\mathbf{x}) - f(\mathbf{y})| < \varepsilon/[3\ \text{area}(R)]$ whenever $\|\mathbf{x} - \mathbf{y}\| < \delta$. We choose N so large that whenever $n \ge N$ the diameter of each subrectangle in the nth regular partition is less than δ. Since $\lim_{n\to\infty} S_n = S$ and $\lim_{n\to\infty} S_n^* = S^*$, we can assume that N has been chosen so large that $n \ge N$ implies that $|S_n - S| < \varepsilon/3$ and $|S_n^* - S^*| < \varepsilon/3$. Also, for $n \ge N$ we know by uniform continuity that if $\mathbf{c}_{jk}$ and $\mathbf{c}_{jk}^*$ are points in the same subrectangle R_{jk} of the nth partition, then $|f(\mathbf{c}_{jk}) - f(\mathbf{c}_{jk}^*)| < \varepsilon/[3\ \text{area}(R)]$. Thus

$$\begin{aligned} |S_n - S_n^*| &= \left| \sum_{j,k} f(\mathbf{c}_{jk}) \Delta x^n \Delta y^n - \sum_{j,k} f(\mathbf{c}_{jk}^*) \Delta x^n \Delta y^n \right| \\ &\le \sum_{j,k} |f(\mathbf{c}_{jk}) - f(\mathbf{c}_{jk}^*)| \Delta x^n \Delta y^n < \frac{\varepsilon}{3}. \end{aligned}$$

We now write

$$|S - S^*| = |S - S_n + S_n - S_n^* + S_n^* - S^*| \le |S - S_n| + |S_n - S_n^*| + |S_n^* - S| < \varepsilon$$

and so the lemma is proved. ■

Putting lemmas 1 and 2 together proves Theorem 1 of §5.2 of the main text:

Theorem 1 of §5.2 of the Text. *Any continuous function defined on a rectangle R is integrable.*

Historical Note. Cauchy presented the first published proof of this theorem in his résumé of 1823, in which he points out the need to prove the existence of the integral as a limit of a sum. In this paper he first treats continuous functions (as we are doing now), but on an interval $[a, b]$. (The proof is essentially the same.) However, his proof was not rigorous, since it lacked the notion of uniform continuity, which was not available at that time.

The notion of a Riemann sum S_n for a function f certainly predates Bernhard Riemann (1826–1866). The sums are probably named after him because he developed a theoretical approach to the study of integration in a fundamental paper on trigonometric series in 1854. His approach, although later generalized by Darboux (1875) and Stieltjes (1894), was to last more than half a century until it was augmented by the theory Lebesque presented to the mathematical world in 1902. This latter approach to integration theory is generally studied in graduate courses in mathematics.

The proof of Theorem 2 of §5.2 is left to the reader in Exercises 4 to 6 at the end of this section. The main ideas are essentially contained in the proof of Theorem 1.

Additivity Theorem. Our next goal will be to present a proof of property (iv) of the integral from §5.2, namely, its additivity. However, because of some technical difficulties in establishing this result in its full generality, we shall prove it only in the case in which f is continuous.

Theorem. Additivity of the Integral. *Let R_1 and R_2 be two disjoint rectangles (rectangles whose intersection contains no rectangle) such that $Q = R_1 \cup R_2$ is again a rectangle as in Figure 5.6.2. If f is a function that is continuous over Q and hence over each R_i, then*

$$\iint_Q f = \iint_{R_1} f + \iint_{R_2} f. \tag{3}$$

Proof. The proof depends on the ideas that have already been presented in the proof of Theorem 1.

The fact that f is integrable over Q, R_1, and R_2 follows from Theorem 1. Thus all three integrals in equation (3) exist, and it is necessary only to establish equality.

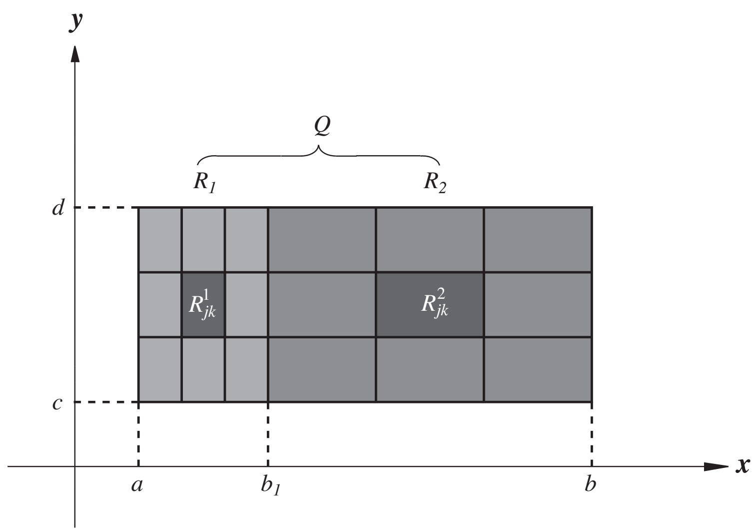


FIGURE 5.6.2. Elements of a regular partition of R_1 and R_2.

Without loss of generality we can assume that

$$R_1 = [a, b_1] \times [c, d] \text{ and } R_2 = [b_1, b] \times [c, d]$$

(see Figure 5.6.2). Again we must develop some notation. Let

$$\Delta x_1^n = \frac{b_1 - a}{n}, \quad \Delta x_2^n = \frac{b - b_1}{n}, \quad \Delta x^n = \frac{b - a}{n}, \quad \text{and} \quad \Delta y^n = \frac{d - c}{n}.$$

Let

$$S_n^1 = \sum_{j,k} f(\mathbf{c}_{jk}^1) \Delta x_1^n \Delta y^n \tag{4}$$

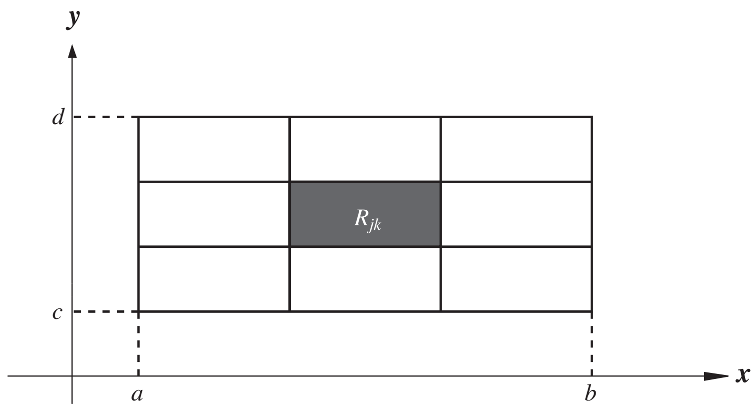
$$S_n^2 = \sum_{j,k} f(\mathbf{c}_{jk}^2) \Delta x_2^n \Delta y^n \tag{5}$$

$$S_n = \sum_{j,k} f(\mathbf{c}_{jk}) \Delta x^n \Delta y^n \tag{6}$$

where $\mathbf{c}_{jk}^1, \mathbf{c}_{jk}^2$, and $\mathbf{c}_{jk}$ are points in the jkth subrectangle of the nth regular partition of R_1, R_2, and Q, respectively. Let $S^i = \lim_{n\to\infty} S_n^i$, where $i = 1, 2$, and $S = \lim_{n\to\infty} S_n$. It must be shown that $S = S^1 + S^2$, which we will accomplish by showing that for arbitrary $\varepsilon > 0, |S - S^1 - S^2| < \varepsilon$.

By the uniform continuity of f on Q we know that given $\varepsilon > 0$ there is a $\delta > 0$ such that whenever $\|\mathbf{x} - \mathbf{y}\| < \delta, |f(\mathbf{x}) - f(\mathbf{y})| < \varepsilon$. Let N be so big that for all $n \geq N, |S_n - S| < \varepsilon/3, |S_n^i - S^i| < \varepsilon/3, i = 1, 2$, and if

$\mathbf{x}, \mathbf{y}$ are any two points in any subrectangle of the nth partition of either R_1, R_2, or Q then $|f(\mathbf{x}) - f(\mathbf{y})| < \varepsilon/[3 \text{ area}(Q)]$. Let us consider the nth regular partition of R_1, R_2, and Q. These form a collection of subrectangles that we shall denote by $R^1_{jk}, R^2_{jk}, R_{jk}$, respectively [see Figures 5.6.2 and 5.6.3(a)].



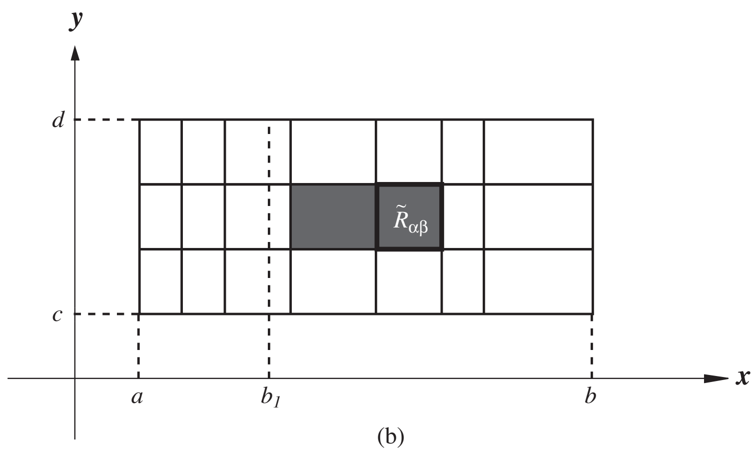


FIGURE 5.6.3. (a) A regular partition of Q. (b) The vertical and horizontal lines of this subdivision are obtained by taking the union of the vertical and horizontal of Figures 5.6.2 and 5.6.3(a).

If we superimpose the subdivision of Q on the nth subdivisions of R_1 and R_2, we get a new collection of rectangles, say $\tilde{R}_{\alpha\beta}, \beta = 1, \ldots, n$ and $\alpha = 1, \ldots, m$, where $m > n$; see Figure 5.6.3(b).

Each $\tilde{R}_{\alpha\beta}$ is contained in some subrectangle R_{jk} of Q and in some subrectangle of the nth partition of either R_1 or R_2. Consider equalities (4), (5), and (6) above. These can be rewritten as

$$S_n^i = \sum_{j,k} \sum_{\tilde{R}_{\alpha\beta}\subset R_i} f(\mathbf{c}_{jk}^i)\,\text{area}\,(\tilde{R}_{\alpha\beta}) = \sum_{\substack{\alpha,\beta\\ \tilde{R}_{\alpha\beta}\subset R_i}} f(\tilde{\mathbf{c}}_{\alpha\beta})\,\text{area}\,(\tilde{R}_{\alpha\beta}).$$

where $\tilde{\mathbf{c}}_{\alpha\beta} = \mathbf{c}_{\alpha\beta}^i$ if $\tilde{R}_{\alpha\beta} \subset R_{jk}^i, i = 1, 2$, and

$$S_n = \sum_{j,k} \sum_{\tilde{R}_{\alpha\beta}\subset R_{jk}} f(\mathbf{c}_{jk})\,\text{area}\,(\tilde{R}_{\alpha\beta}) = \sum_{\alpha,\beta} f(\mathbf{c}_{\alpha\beta}^*)\,\text{area}\,(\tilde{R}_{\alpha\beta}).$$

where $\mathbf{c}_{\alpha,\beta}^* = \mathbf{c}_{jk}$ if $\tilde{R}_{\alpha\beta} \subset R_{jk}$.

For the reader encountering such index notation for the first time, we point out that

$$\sum_{\substack{\alpha,\beta\\ \tilde{R}_{\alpha\beta}\subset R_i}}$$

means that the summation is taken over those α's and β's such that the corresponding rectangle $\tilde{R}_{\alpha\beta}$ is contained in the rectangle R_i.

Now the sum for S_n can be split into two parts:

$$S_n = \sum_{\substack{\alpha,\beta\\ \tilde{R}_{\alpha\beta}\subset R_1}} f(\mathbf{c}_{\alpha\beta}^*)\,\text{area}\,(\tilde{R}_{\alpha\beta}) + \sum_{\substack{\alpha,\beta\\ \tilde{R}_{\alpha\beta}\subset R_2}} f(\mathbf{c}_{\alpha\beta}^*)\,\text{area}\,(\tilde{R}_{\alpha\beta}).$$

From these representations and the triangle inequality, it follows that

$$\begin{aligned}|S_n - S_n^1 - S_n^2| &\le \left| \sum_{\substack{\alpha,\beta\\ \tilde{R}_{\alpha\beta}\subset R_1}} [f(\mathbf{c}_{\alpha\beta}^*) - f(\tilde{\mathbf{c}}_{\alpha\beta})]\,\text{area}(\tilde{R}_{\alpha\beta}) \right| \\ &\quad + \left| \sum_{\substack{\alpha,\beta\\ \tilde{R}_{\alpha\beta}\subset R_2}} [f(\mathbf{c}_{\alpha\beta}^*) - f(\tilde{\mathbf{c}}_{\alpha\beta})]\,\text{area}(\tilde{R}_{\alpha\beta}) \right| \\ &\le \frac{\varepsilon}{3\,\text{area}\,Q} \sum_{\substack{\alpha,\beta\\ \tilde{R}_{\alpha\beta}\subset R_1}} \text{area}\,(\tilde{R}_{\alpha\beta}) \\ &\quad + \frac{\varepsilon}{3\,\text{area}\,Q} \sum_{\substack{\alpha,\beta\\ \tilde{R}_{\alpha\beta}\subset R_2}} \text{area}\,(\tilde{R}_{\alpha\beta}) < \frac{\varepsilon}{3}.\end{aligned}$$

In this step we used the uniform continuity of f. Thus $|S_n - S_n^1 - S_n^2| < \varepsilon/3$ for $\ge N$. But

$$|S - S_n| < \frac{\varepsilon}{3}, \quad |S_n^1 - S^1| < \frac{\varepsilon}{3} \quad \text{and} \quad |S_n^2 - S^2| < \frac{\varepsilon}{3}.$$

As in Lemma 2, an application of the triangle inequality shows that $|S - S^1 - S^2| < \varepsilon$, which completes the proof. ■

Example 6. *Let C be the graph of a continuous function $\phi : [a, b] \to \mathbb{R}$. Let $\epsilon > 0$ be any positive number. Show that C can be placed in a finite union of boxes $B_i = [a_i, b_i] \times [c_i, d_i]$ such that C does not contain a boundary point of $\cup B_i$ and such that $\sum \text{area}(B_i) \leq \epsilon$.*

Solution. Let $\epsilon > 0$. Since f is uniformly continuous there exists a δ, with $0 < \delta < 1$ such that if $|x - \omega| < \delta$, then $|f(x) - f(\omega)| < \epsilon/32$. (We see why the "32" as we proceed) — of course in reality one does a sketch of the idea first and through a trial run one "discovers" that the right number to put here is indeed 32. Let $n > 1/\delta$ and subdivide the interval $[a, b]$ into $2n$ equal parts with $x_0 < x_1 < \ldots < x_{2n}$ the corresponding partition. Let B_i be the rectangle centered at $(x_i, f(x_i))$ with width $1/n$ and height $\epsilon/4$. The area of each rectangle is $\epsilon/4n$. There are $(2n + 1)$ such rectangles for a total area of

$$\left(\frac{\epsilon}{4n}\right)(2n + 1) = \frac{\epsilon}{2} + \frac{\epsilon}{2n} < \epsilon.$$

It remains only to check that C does not contain a boundary point of $\cup_i B_i$. If the graph touches the top edge of some Box B_i, then there is some $(x, f(x)) \epsilon B_i$ such that $|f(x) - f(x_i)| \geq \epsilon/8$ ($\epsilon/8$ is half the height of B_i). But by uniform continuity $|f(x) - f(x_i)| < \epsilon/32$ a contradiction.

Similarly if C intersects a vertical side of B_i it must do so in a portion which is in the complement of $B_{i-1} \cup B_{i+1}$. This again contradicts uniform continuity. ♦

Exercises.

1. Show that if a and b are two numbers such that for any $\varepsilon > 0$, $|a-b| < \varepsilon$, then $a = b$.

2. (a) Let f be the function on the half-open interval $(0, 1]$ defined by $f(x) = 1/x$. Show that f is continuous at every point of $(0, 1]$ but not uniformly continuous.

   (b) Generalize this example to $\mathbb{R}^2$.

3. Let R be the rectangle $[a, b] \times [c, d]$ and f be a bounded function that is integrable over R.

   (a) Show that f is integrable over $[(a + b)/2, b] \times [c, d]$.

   (b) Let N be any positive integer. Show that f is integrable over $[(a + b)/N, b] \times [c, d]$.

Exercises 4 to 6 are intended to give a proof of Theorem 2 of §5.2.

4. Let C be the graph of a continuous function $\phi : [a, b] \to \mathbb{R}$. Let $\varepsilon > 0$ be any positive number. Show that C can be placed in a finite union of boxes $B_i = [a_i, b_i] \times [c_i, d_i]$ such that C does not contain a boundary point of $\cup B_i$ and such that Σ area $(B_i) \leq \varepsilon$. (Hint: Use the uniform continuity principle presented in this section.)

5. Let R and B be rectangles and let $B \subset R$. Consider the nth regular partition of R and let b_n be the sum of the areas of all rectangles in the partition that have a nonempty intersection with B. Show that $\lim_{n\to\infty} b_n = \text{area}(B)$.

6. Let R be a rectangle and $C \subset R$ the graph of a continuous function ϕ. Suppose that $f : R \to \mathbb{R}$ is bounded and continuous except on C. Use Exercises 4 and 5 above and the techniques used in the proof of Theorem 1 of this section to show that f is integrable over R.

7. (a) Use the uniform continuity principle to show that if $\phi : [a, b] \to \mathbb{R}$ is a continuous function, then ϕ is bounded.
   (b) Generalize part (a) to show that a given continuous function $f : [a, b] \times [c, d] \to \mathbb{R}$ is bounded.
   (c) Generalize part (b) still further to show that if $f : D \to \mathbb{R}$ is a continuous function on a closed and bounded set $D \subset \mathbb{R}^n$, then f is bounded.

6
Integrals over Curves and Surfaces

To complete his Habilitation, ***Riemann*** had to give a lecture. He prepared three lectures, two on electricity and one on geometry. Gauss had to choose one of the three for Riemann to deliver and, against Riemann's expectations, Gauss (his advisor) chose the lecture on geometry. Riemann's lecture *Über die Hypothesen welche der Geometrie zu Grunde liegen* (On the hypotheses that lie at the foundations of geometry), delivered on 10 June 1854, became a classic of mathematics.

There were two parts to Riemann's lecture. In the first part he posed the problem of how to define an n-dimensional space and ended up giving a definition of what today we call a Riemannian space. Freudenthal writes:

> *It possesses shortest lines, now called geodesics, which resemble ordinary straight lines. In fact, at first approximation in a geodesic coordinate system such a metric is flat Euclidean, in the same way that a curved surface up to higher-order terms looks like its tangent plane. Beings living on the surface may discover the curvature of their world and compute it at any point as a consequence of observed deviations from Pythagoras' theorem.*

In fact the main point of this part of Riemann's lecture was the definition of the curvature tensor . The second part of Riemann's lecture posed deep questions about the relationship of geometry to the world we live in. He asked what the dimension of real space was and what geometry described real space. The lecture was too far ahead of its time to be appreciated by most scientists of that time. Monastyrsky writes:

Among Riemann's audience, only Gauss was able to appreciate the depth of Riemann's thoughts. ... The lecture exceeded all his expectations and greatly surprised him. Returning to the faculty meeting, he spoke with the greatest praise and rare enthusiasm to Wilhelm Weber about the depth of the thoughts that Riemann had presented.

It was not fully understood until sixty years later. Freudenthal writes:

The general theory of relativity splendidly justified his work. In the mathematical apparatus developed from Riemann's address, Einstein found the frame to fit his physical ideas, his cosmology , and cosmogony : and the spirit of Riemann's address was just what physics needed: the metric structure determined by data.

From the Riemann website
http://www-gap.dcs.st-and.ac.uk/~history/
Mathematicians/Riemann.html

Supplement 6.2A
A Challenging Example

Example. Evaluate

$$\iint_R \sqrt{x^2+y^2}\,dxdy,$$

where $R = [0,1] \times [0,1]$.

Solution. This double integral is equal to the volume of the region under the graph of the function

$$f(x,y) = \sqrt{x^2+y^2}$$

over the rectangle $R = [0,1] \times [0,1]$; that is, it equals the volume of the three-dimensional region shown in Figure 6.1.1.

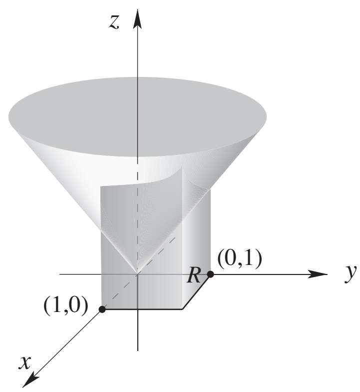


FIGURE 6.1.1. Volume of the region under $z = \sqrt{x^2 + y^2}$ and over $R = [0,1] \times [0,1]$.

As it stands, this integral is difficult to evaluate. Since the integrand is a simple function of $r^2 = x^2 + y^2$, we might try a change of variables to polar coordinates. This will result in a simplification of the integrand but, unfortunately, not in the domain of the integration. However, the simplification is sufficient to enable us to evaluate the integral. To apply Theorem 2 of the text with polar coordinates, refer to Figure 6.1.2.

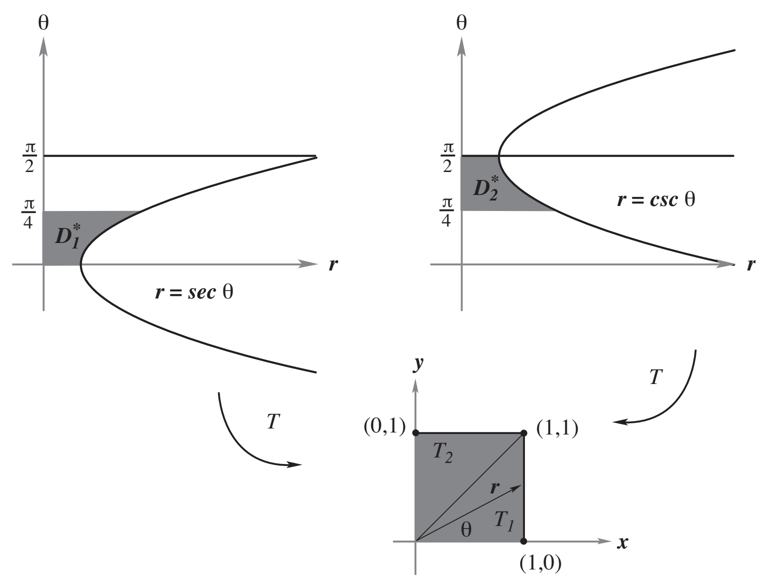


FIGURE 6.1.2. The polar-coordinate transformation takes D_1^* to the triangle T_1 and D_2^* to T_2.

The reader can verify that R is the image under $T(r,\theta) = (r\cos\theta, r\sin\theta)$ of the region $D^* = D_1^* \cup D_2^*$ where for D_1^* we have $0 \le \theta \le \frac{1}{4}\pi$ and $0 \le r \le \sec\theta$; for D_2^* we have $\frac{1}{4}\pi \le \theta \le \frac{1}{2}\pi$ and $0 \le r \le \csc\theta$. The transformation T sends D_1^* onto a triangle T_1 and D_2^* onto a triangle T_2. The transformation T is one-to-one except when $r = 0$, and so we can apply Theorem 2. From the symmetry of $z = \sqrt{x^2+y^2}$ on R, we can see that

$$\iint_R \sqrt{x^2+y^2}\; dxdy = 2\iint_{T_1} \sqrt{x^2+y^2}\; dxdy.$$

Changing to polar coordinates, we obtain

$$\iint_{T_1} \sqrt{x^2+y^2}\; dxdy = \iint_{D_1^*} \sqrt{r^2}r\; drd\theta = \iint_{D_1^*} r^2\; drd\theta.$$

Next we use iterated integration to obtain

$$\iint_{D_1^*} r^2\; drd\theta = \int_0^{\pi/4}\left[\int_0^{\sec\theta} r^2\; dr\right]d\theta = \frac{1}{3}\int_0^{\pi/4} \sec^3\theta\; d\theta.$$

Consulting a table of integrals (see the back of the book) to find $\int \sec^3 x\; dx$, we have

$$\int_0^{\pi/4} \sec^3\theta\; d\theta = \left[\frac{\sec\theta\tan\theta}{2}\right]_0^{\pi/4} + \frac{1}{2}\int_0^{\pi/4} \sec\theta\; d\theta = \frac{\sqrt{2}}{2} + \frac{1}{2}\int_0^{\pi/4} \sec^3\theta\; d\theta.$$

Consulting the table again for $\int \sec x\; dx$, we find

$$\frac{1}{2}\int_0^{\pi/4} \sec\theta\; d\theta = \frac{1}{2}[\log|\sec\theta + \tan\theta|]_0^{\pi/4} = \frac{1}{2}\log(1+\sqrt{2}).$$

Combining these results and recalling the factor $\frac{1}{3}$, we obtain

$$\iint_{D_1^*} r^2\; drd\theta = \frac{1}{3}\left[\frac{\sqrt{2}}{2} + \frac{1}{2}\log(1+\sqrt{2})\right] = \frac{1}{6}[\sqrt{2} + \log(1+\sqrt{2})].$$

Multiplying by 2, we obtain the answer

$$\iint_R \sqrt{x^2+y^2}\; dxdy = \frac{1}{3}[\sqrt{2} + \log(1+\sqrt{2})].$$ ♦

Instead of using the substitution T, one can alternatively divide the original square into two triangles T_1 and T_2 as in the text, write the integral over T_1 as a double integral (first with respect to y, then with respect to x); in the integral over y, substitute $y = xv$, then use the standard integral number 43 at the back of the book.

Here is another example of the use of the change of variables theorem.

Exercise 15 *Evaluate*

$$\iint_B \exp\left(\frac{y-x}{y+x}\right) dx\,dy$$

where B is the inside of the triangle with vertices at $(0,0),(0,1)$ *and* $(1,0)$.

Solution. Using the change of variables $u = y - x, v = y + x$, we get $|\partial(x,y)/\partial(u,v)| = 1/2$. Thus,

$$\begin{aligned}
\iint_B \exp\left(\frac{y-x}{y+x}\right) dx\,dy &= \iint_{B'} \exp\left(\frac{u}{v}\right)\frac{1}{2} du\,dv \\
&= \frac{1}{2}\int_0^1 \int_{-v}^{v} \exp\left(\frac{u}{v}\right) du\,dv \\
&= \frac{1}{2}\int_0^1 \left(v\exp\left(\frac{u}{v}\right)\Big|_{-v}^{v}\right) dv \\
&= \frac{1}{2}\int_0^1 v\left(e - e^{-1}\right) dv = \frac{1}{4}(e - e^{-1}). \quad \blacklozenge
\end{aligned}$$

Supplement 6.2B
The Gaussian Integral

The purpose of this supplement is to prove the equality of the following two limits

$$\lim_{a\to\infty} \iint_{D_a} e^{-(x^2+y^2)} dx\,dy = \lim_{a\to\infty} \iint_{R_a} e^{-(x^2+y^2)} dx\,dy,$$

which was used in the derivation of the Gaussian integral formula. In the text, we showed that we showed that

$$\lim_{a\to\infty} \iint_{D_a} e^{-(x^2+y^2)} dx\,dy$$

exists by directly evaluating it. Thus, it suffices to show that

$$\lim_{a\to\infty} \left(\iint_{R_a} e^{-(x^2+y^2)} dx\,dy - \iint_{D_a} e^{-(x^2+y^2)} dx\,dy \right)$$

equals zero. The limit equals

$$\lim_{a\to\infty} \iint_{C_a} e^{-(x^2+y^2)} dx\,dy,$$

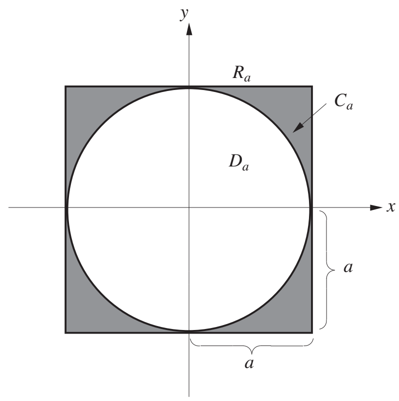


FIGURE 6.1.3. The region C_a lies between the square R_a and the circle D_a.

where C_a is the region between R_a and D_a (see Figure 6.1.3).

In the region C_a, $\sqrt{x^2+y^2} \geq a$ (the radius of D_a), so $e^{-(x^2+y^2)} \leq e^{-a^2}$. Thus,

$$0 \leq \iint_{C_a} e^{-(x^2+y^2)} dx\, dy \leq \iint_{C_a} e^{-a^2} dx\, dy$$

$$= e^{-a_2} \text{ area } (C_a) = e^{-a^2}(4a^2 - \pi a^2) = (4-\pi)a^2 e^{-a^2}.$$

Thus it is enough to show that $\lim_{a\to\infty} a^2 e^{-a^2} = 0$. But, by l'Hôpital's rule

$$\lim_{a\to\infty} a^2 e^{-a^2} = \lim_{a\to\infty} \left(\frac{a^2}{e^{a^2}}\right) = \lim_{a\to\infty} \left(\frac{2a}{2ae^{a^2}}\right) = \lim_{a\to\infty} \frac{1}{e^{a^2}} = 0,$$

as required. ■

7
Integrals Over Paths and Surfaces

Supplement for §7.4
The Problem of Plateau

We end this section by describing the fascinating classic area problem of Plateau, which has enjoyed a long history in mathematics. The Belgian physicist Joseph Plateau (1801–1883) carried out many experiments from 1830 to 1869 on surface tension and capillary phenomena, experiments that had enormous impact at the time and were repeated by notable nineteenth-century physicists, such as Michael Faraday (1791–1867). The corresponding collection of mathematical problems relating to soap films was named in 1904 after Plateau by the great French mathematician Henri Lebesgue (1875–1941).

If a wire is dipped into a soap or glycerine solution, then one usually withdraws a soap film spanning the wire. Some examples are given in Figure 7.4.1, although readers might like to perform the experiment for themselves. Plateau raised the mathematical question: For a given boundary (wire), how does one prove the existence of such a surface (soap film) and how many surfaces can there be? The underlying physical principle is that nature tends to minimize area; that is, the surface that forms should be a surface of least area among all possible surfaces that have the given curve as their boundary. This again is another example of the action principle of Maupertuis and Leibniz (c.f. Section .3.3)

For soap film surfaces that are disklike, the problem can be formulated in the following way. Let $D \subset \mathbb{R}^2$ be the unit disk defined to be the set

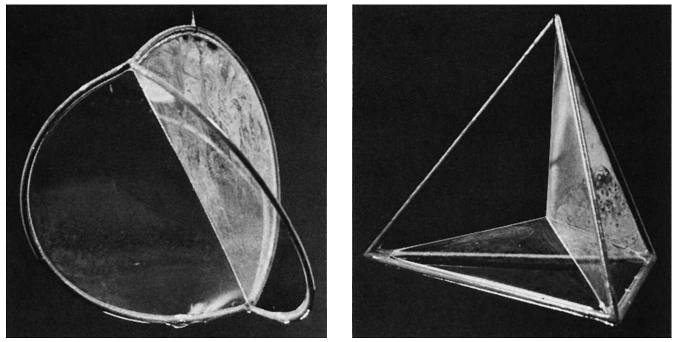


FIGURE 7.4.1. Two soap films spanning wires.

$\{(x, y) \mid x^2 + y^2 \leq 1\}$ and let ∂D be its boundary. Furthermore, suppose that the image Γ of $\mathbf{c}\colon [0,\, 2\pi] \to \mathbb{R}^3$ is a simple closed curve, Γ representing a wire in $\mathbb{R}^3$.

Let ßS be the set of all maps $\mathbf{\Phi}\colon\ D \to \mathbb{R}^3$ such that $\mathbf{\Phi}(\partial D) = \Gamma, \mathbf{\Phi}$ is of class C^1, and $\mathbf{\Phi}$ is one-to-one on ∂D. Each $\mathbf{\Phi} \in$ ßS represents a parametric C^1 "disklike" surface "spanning" the wire Γ.

The soap films in Figure 7.4.1 are not disklike, but represent a system of multiple disklike surfaces. Figure 7.4.2 shows a contour that bounds two disklike surfaces and one nondisklike surface.

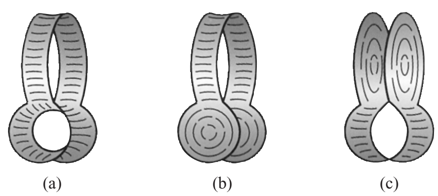


FIGURE 7.4.2. Soap film surfaces; (b) and (c) are disklike surfaces, but (a) is not.

For each $\mathbf{\Phi} \in$ ßS, consider the area of the image surface, namely, $A(\mathbf{\Phi}) = \iint_D \|\mathbf{T}_u \times \mathbf{T}_v\|\, du\, dv$. This area is a function that assigns to each parametric surface its area. Plateau asked whether A has a minimum on ßS; that is, does there exist a $\mathbf{\Phi}_0$ such that $A(\mathbf{\Phi}_0) \leq A(\mathbf{\Phi})$ for all $\mathbf{\Phi} \in$ ßS? Unfortunately, the methods of this book are not adequate to solve this problem. We can tackle questions of finding minima of real-valued functions of several variables, but in no way can the set ßS be thought of as a region in $\mathbb{R}^n$ for *any n*!

In his own study of surfaces of least area, Weierstrass showed that if a minimum

$$\mathbf{\Phi}_0(u, v) = (x(u, v), y(u, v), z\,(u, v))$$

existed at all, it would have to satisfy (after suitable normalizations) the partial differential equations

(i) $\nabla^2 \mathbf{\Phi}_0 = 0$

(ii) $\dfrac{\partial \mathbf{\Phi}_0}{\partial u} \cdot \dfrac{\partial \mathbf{\Phi}_0}{\partial v} = 0$

(iii) $\left\| \dfrac{\partial \mathbf{\Phi}_0}{\partial u} \right\| = \left\| \dfrac{\partial \mathbf{\Phi}_0}{\partial v} \right\|$

where $\|\mathbf{w}\|$ denotes the "norm" or length of the vector $\mathbf{w}$. This example illustrates the intimate connections between problems of maxima and minima (the calculus of variations) and the subject of partial differential equations.

For well over 70 years, mathematicians such as Riemann, Weierstrass, H. A. Schwarz, Darboux, and Lebesgue puzzled over the challenge posed by Plateau. In 1931 the question was finally settled when Jesse Douglas showed that such a $\mathbf{\Phi}_0$ existed. However, many questions about soap films remain unsolved, and this area of research is still active today.[1]

[1]For more information on this fascinating subject, the reader may consult *The Parsimonious Universe*: *Shape and Form in the Natural World*, by S. Hildebrandt and A. Tromba, Springer-Verlag, New York/Heidelberg, 1995.

8
The Integral Theorems of Vector Analysis

George Green (1793–1841), a self-taught English mathematician, undertook to treat static electricity and magnetism in a thoroughly mathematical fashion. In 1828 Green published a privately printed booklet, *An essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism.* This was neglected until Sir William Thomson (Lord Kelvin, 1824–1907) discovered it, recognized its great value, and had it published in the *Journal für Mathematik* starting in 1850. Green, who learned much from Poisson's papers, also carried over the notion of the potential function to electricity and magnetism.

Morris Klein
Mathematical Thought from Ancient to Modern Times

Supplement for §8.2 Reorienting Astronauts

Reorienting Astronauts Another example to help visualize this effect is to consider astronauts who wish to reorient themselves in a free-space environment. As with the falling cat, this motion can again be achieved using internal gyrations, or *shape changes.* For instance, consider astronauts moving their arms much like the motion of arms stirring liquid in a large kettle. The arms are held out forward, to lie in a horizontal plane that goes through the shoulders, parallel to the floor; the hands are clasped together and remain in this horizontal plane during the circular stirring motion. At the point of maximum extension of the arms, the inertia of the body about a vertical axis is also at a maximum. Conservation of angular momentum requires that the body rotate in an opposite and proportional manner to the motion of the arms. As the arms rotate around and are brought in, however, the inertia of the body is reduced. The motion of the body in reaction is therefore also reduced. Thus, in one complete cycle of arm movement, the body undergoes a net rotation opposite the direction of arm motion. When the desired orientation is achieved, the astronaut needs merely to stop the arm motion in order to come to rest. One often refers to the extra motion that is achieved as ***geometric phase***.

Link with Non-Euclidean Geometry The theory of geometric phases also shows up in an interesting way in non-Euclidean geometry—as in the geometry of triangles drawn on a sphere. A simple way to explain this link is as follows. Hold your hand at arm's length, but allow rotation in your shoulder joints. Move your hand along three great circles, forming a triangle on the sphere; during the motion along each arc, always keep your thumb *parallel*; that is, it should move in such a way that it forms a *fixed* angle with the direction of motion along each arc and does not rotate when switching arcs. After completing the circuit around the triangle, your thumb will return rotated through an angle relative to its starting position (see Figure 8.2.13). Can you see in Figure 8.2.13 that the angle of rotation is 90° (or $\pi/2$ radians) and that this is what happens when you do the thumb experiment yourself?

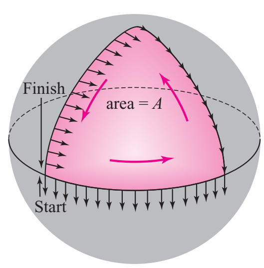


FIGURE 8.0.1. A parallel movement of your thumb around a spherical triangle produces a phase shift.

For general spherical triangles, this angle (in radians) is given by $\Theta = \Delta - \pi$, where Δ is the sum of the angles of the triangle. The fact that Θ is strictly positive (!) is one of the basic truths of non-Euclidean geometry—the sum of the angles of a right triangle on a sphere is greater than π! This angle is also related to the *area* A enclosed by the triangle through the relation $\Theta = A/r^2$, where r is the radius of the sphere. The rotational shift of the thumb during the course of its cyclic journey around the spherical triangle is directly related to the curvature of the sphere and to the area enclosed by the path that is traced out. Notice first that for a spherical triangle that is 1/8 of the sphere, $A = 4\pi r^2/8 = \pi r^2/2$. Thus, $A/r^2 = \pi/2$. Notice also that when $r \to \infty$, the sphere becomes flatter and thus approaches a Euclidean plane, in which case $\Theta = 0$.

The cyclic journey of the thumb around the closed path is analogous to the cyclic internal motions made by the cat during its fall; the 90° shift in the direction of the thumb after one trip around is analogous to the 180° reorientation of the cat. A deeper look at the underlying mathematics shows that, in fact, they are both instances of the same phenomenon (called *holonomy*)—and Stokes' theorem is the key to understanding it.

Supplement for §8.3 Exact Differentials

The main theoretical point is that, to solve a differential equation in the variables (x, y) of the form

$$P(x,y) + Q(x,y)\frac{dy}{dx} = 0, \tag{8.2.1}$$

we proceed as follows: first we test to see if $P dx + Q dy$ is ***exact***; that is, if the corresponding vector field $P\mathbf{i} + Q\mathbf{j}$ is conservative. If it is, then there is a function $f(x, y)$ such that

$$P = \frac{\partial f}{\partial x}; \quad Q = \frac{\partial f}{\partial y}.$$

Recall that the test for exactness is

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

and that this corresponds to equality of mixed partial derivatives of f.

If the equation is exact and we find a corresponding f, then the equation $f(x, y) =$ constant implicitly defines solutions as level sets of f. This assertion is readily checked by implicit differentiation as follows. If x and y are each functions of t and lie on the surface constant $= f(x, y)$, then differentiating both sides and using the chain rule gives

$$0 = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = P\frac{dx}{dt} + Q\frac{dy}{dt}.$$

In fact, this is one possible way to interpret equation (8.2.1). If y is a function of x and we similarly differentiate the equation constant $= f(x, y)$

with respect to x and use the chain rule in a similar way, then we literally get equation (8.2.1).

Example 1. Solve the following differential equation satisfying the given conditions:

$$\cos y \sin x + \sin y \cos x \frac{dy}{dx} = 0, \quad y\left(\frac{\pi}{4}\right) = 0.$$

Solution. The equation is exact, since

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(\cos y \sin x) = -\sin y \sin x,$$

and

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(\sin y \cos x) = -\sin y \cos x.$$

The function $f(x, y)$ such that $P = \partial f/\partial x$ and $Q = \partial f/\partial y$ can then be found by integration:

$$\int P\, dx = \int \cos y \sin x\, dx = -\cos y \cos x + g(y) + C_1$$

$$\int Q\, dy = \int \sin y \cos x\, dy = -\cos y \cos x + h(x) + C_2,$$

where $g(y)$ is a function of y only (which will show up when integrating Q with respect to y), $h(x)$ is a function of x only (which will show up when integrating P with respect to x), and C_2 and C_2 are two constants. Compare those two results, we find that $f(x, y) = -\cos y \cos x = C$, a constant. Since $y = 0$ when $x = \pi/4$, we find that $C = -\cos 0 \cdot \cos(\pi/4) = -1/\sqrt{2}$, thus, the solution is defined by $\cos y \cos x = 1/\sqrt{2}$. ♦

Sometimes multiplying an equation by an appropriate factor can make it exact. Such factors are called ***integrating factors***.

Example 2. Solve the equation

$$x\frac{dy}{dx} = xy^2 + y$$

by using the integrating factor $1/y^2$.

Solution. The integrating factor $1/y^2$ makes our equation exact. To check this, first multiply and then check for exactness: The equation is

$$\frac{x}{y^2}\frac{dy}{dx} - x - \frac{1}{y} = 0.$$

Now,

$$\frac{\partial}{\partial y}\left(-x-\frac{1}{y}\right)=\frac{1}{y^2};$$

and

$$\frac{\partial}{\partial x}\left(\frac{x}{y^2}\right)=\frac{1}{y^2}.$$

Thus, the equation is exact. To find an antiderivative, write

$$\int\left(-x-\frac{1}{y}\right)dx=-\frac{x^2}{2}-\frac{x}{y}+g_1(y), \tag{8.2.2}$$

$$\int\frac{x}{y^2}dy=-\frac{x}{y}+g_2(x). \tag{8.2.3}$$

Comparing (8.2.2) and (8.2.3), we see that if $g_2(x)=-x^2/2$ and $g_1(y)=C$, a constant, then the equation is solved, and so the solution is defined by the level curves of f, *i.e.*,

$$f(x,y)=\frac{x}{y}+\frac{x^2}{2}=C. \qquad \blacklozenge$$

Supplement for §8.5
Green's Functions

Some Differential Equations of Mechanics and Technology

Isaac Newton reputedly said, "All in nature reduces to differential equations." This point of view was paraphrased by Max Planck (see the Historical Note in Section 3.3): "... Present day physics, as far as it is theoretically organized, is completely governed by a system of space–time differential equations."

In this section, we apply the central theorems of vector analysis to the derivation of the differential equations governing heat transfer, electromagnetism, and the motion of some fluids.

Keep in mind the importance of these problems in modern technology. For example, a good understanding of fluids and the ability to do computations to solve their governing equations is at the heart of how one builds a modern airplane or designs a submarine. For instance, the flow of air (the fluid in this case) over the wings of an aircraft is very subtle, even though the governing equations are relatively simple. We shall derive a slightly idealized form of these equations in this section. Likewise, the equations of electromagnetism, as we will discuss in the following paragraphs, is central to the communications industry; wireless, television, and much of the operation of modern electronic devices, including computers, depends on these and related fundamental equations.

Conservation Laws

As preparation for deriving the equations of a fluid, let us first discuss an important equation that is referred to as a *conservation* equation. For fluids, it expresses the conservation of mass; for electromagnetic theory, it expresses the conservation of charge. We shall apply these ideas to the equation for heat conduction and to electromagnetism.

Let $\mathbf{V}(t, x, y, z)$ be a C^1 vector field on $\mathbb{R}^3$ for each t and let $\rho(t, x, y, z)$ be a C^1 real-valued function. By the *law of conservation of mass* for $\mathbf{V}$ and ρ, we mean that the condition

$$\frac{d}{dt}\iiint_W \rho dV = -\iint_{\partial W} \mathbf{J}\cdot\mathbf{n}\, dS$$

holds for all regions W in $\mathbb{R}^3$, where $\mathbf{J} = \rho\mathbf{V}$ (see Figure 8.5.1).

If we think of ρ as a mass density (ρ could also be a charge density)—that is, the mass per unit volume—and of $\mathbf{V}$ as the velocity field of a fluid,

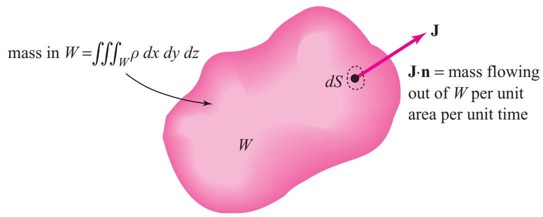


FIGURE 8.5.1. The rate of change of mass in W equals the rate at which mass crosses ∂W.

the condition simply says that the rate of change of total mass in W equals the rate at which mass flows *into* W. Recall that $\iint_{\partial W} \mathbf{J} \cdot \mathbf{n}\, dS$ is called the *flux* of $\mathbf{J}$. We need the following result.

8.5.1 Theorem. *For* $\mathbf{V}$ *and* ρ *(a smooth vector field and a scalar field on* $\mathbb{R}^3$*), the law of conservation of mass for* $\mathbf{V}$ *and* ρ *is equivalent to the condition*

$$\mathit{div}\ \mathbf{J} + \frac{\partial \rho}{\partial t} = 0. \tag{8.5.1}$$

That is,

$$\rho\ \mathit{div}\ \mathbf{V} + \mathbf{V} \cdot \nabla \rho + \frac{\partial \rho}{\partial t} = 0. \tag{1$'$}$$

Here, div $\mathbf{J}$ *means that we compute div* $\mathbf{J}$ *for* t *held fixed, and* $\partial\rho/\partial t$ *means we differentiate* ρ *with respect to* t *for* x, y, z *fixed.*

Proof. First, observe that by differentiating under the integral, we get

$$\frac{d}{dt} \iiint_W \rho\, dx\, dy\, dz = \iiint_W \frac{\partial \rho}{\partial t}\, dx\, dy\, dz$$

and also

$$\iint_{\partial W} \mathbf{J} \cdot \mathbf{n}\ dS = \iiint_W \ \text{div}\ \mathbf{J}\ d\mathbf{V}$$

by the divergence theorem. Thus, conservation of mass is equivalent to the condition

$$\iiint_W \left(\text{div}\ \mathbf{J} + \frac{\partial \rho}{\partial t} \right) dx\, dy\, dz = 0.$$

Because this is to hold for all regions W, it is equivalent to div $\mathbf{J} + \partial\rho/\partial t = 0$. ■

The equation div $\mathbf{J} + \partial\rho/\partial t = 0$ is called the *equation of continuity*. An interesting remark is that using the change of variables formula, the law of conservation of mass may be shown to be equivalent to the condition

$$\frac{d}{dt} \iiint_{W_t} \rho\ dV = 0,$$

where W_t is the image of W obtained by moving each point in W along flow lines of $\mathbf{V}$ for time t. This result is a special case of the *transport theorem* that we discuss next.

The Transport Theorem

The transport theorem is an interesting application of the divergence theorem that will be needed in our derivation of the equations of a fluid.

8.5.2 Theorem. *Let $\mathbf{F}$ be a vector field on $\mathbb{R}^3$ and denote the flow line of $\mathbf{F}$ start- ing at $\mathbf{x}$ after time t by $\phi(\mathbf{x}, t)$. (See the Internet supplement to Section 4.4 for more information.) Let $J(\mathbf{x}, t)$ be the Jacobian of the map ϕ_t: $\mathbf{x} \mapsto \phi(\mathbf{x}, t)$ for t fixed. Then*

$$\frac{\partial}{\partial t} J(\mathbf{x}, t) = [div\ \mathbf{F}(\phi(\mathbf{x}, t))] J(\mathbf{x}, t).$$

For a given function $f(x, y, z, t)$ and a region $W \subset \mathbb{R}^3$, the transport equation holds:

$$\frac{d}{dt} \iiint_{W_t} f(x, y, z, t)\, dx\, dy\, dz = \iiint_{W_t} \left(\frac{Df}{Dt} + f\ \ div\ \mathbf{F} \right) dx\, dy\, dz,$$

where $W_t = \phi_t(W)$, which is the region moving with the flow, and where

$$\frac{Df}{Dt} = \partial f / \partial t + \nabla f \cdot \mathbf{F}$$

is the material derivative.

Taking $f = 1$, Theorem 12 implies that the following assertions are equivalent (which justifies the use of the term *incompressible*):

1. div $\mathbf{F} = 0$
2. volume $(W_t) =$ volume (W)
3. $J(\mathbf{x}, t) = 1$

Let ϕ, J, $\mathbf{F}$, f be as just defined. There is also a vector form of the transport theorem, namely,

$$\frac{d}{dt} \iiint_{W_t} (f\mathbf{F})\, dx\, dy\, dz$$
$$= \iiint_{W_t} \left[\frac{\partial}{\partial t}(f\mathbf{F}) + \mathbf{F} \cdot \nabla (f\mathbf{F}) + (f\mathbf{F}) \text{ div } \mathbf{F} \right] dx\, dy\, dz,$$

where $\mathbf{F} \cdot \nabla(f\mathbf{F})$ denotes the 3×3 derivative matrix $\mathbf{D}(f\mathbf{F})$ operating on the column vector $\mathbf{F}$; in Cartesian coordinates, $\mathbf{F} \cdot \nabla\mathbf{G}$ is the vector whose ith component is

$$\sum_{j=1}^{3} F_j \frac{\partial G^i}{\partial x_j} = F_1 \frac{\partial G^i}{\partial x} + F_2 \frac{\partial G^i}{\partial y} + F_3 \frac{\partial G^i}{\partial z}.$$

We shall leave the proofs of these results, which are extensions of the arguments used to prove Theorem 11, to the reader (see the exercises).

Derivation of Euler's Equation of a Perfect Fluid

The continuity equation is not sufficient to completely determine the motion of a fluid—we need other conditions.

The fluids that the continuity equation governs can be compressible. If div $\mathbf{V} = 0$ (incompressible case) and ρ is constant, equation $(1')$ follows automatically. But in general, even for incompressible fluids, the equation is not automatic, because ρ can depend on (x, y, z) and t. Thus, even if the equation div $\mathbf{V} = 0$ holds, div $(\rho\mathbf{V}) \neq 0$ may still be true.

Here we discuss Euler's equation for a perfect fluid. Consider a nonviscous fluid moving in space with a velocity field $\mathbf{V}$. When we say that the fluid is *perfect*, we mean that if W is any portion of the fluid, forces of pressure act on the boundary of W along its normal. We assume that the force per unit area acting on ∂W is $-p\mathbf{n}$, where $p(x,\ y,\ z,\ t)$ is some function called the *pressure* (see Figure 8.5.2). Thus, the total pressure force acting on W is

$$\mathbf{F}_{\partial W} = \text{ force } = -\iint_{\partial W} p\mathbf{n}\ dS.$$

This is a *vector* quantity; the ith component of $\mathbf{F}_{\partial W}$ is the integral of the ith component of $p\mathbf{n}$ over the surface ∂W (this is therefore the surface integral of a real-valued function). If $\mathbf{e}$ is any fixed vector in space, we have

$$\mathbf{F}_{\partial W} \cdot \mathbf{e} = -\iint_{\partial W} p\mathbf{e} \cdot \mathbf{n}\, dS,$$

which is the integral of a scalar over ∂W. By the divergence theorem and identity (7) in the table of vector identities (Section 4.4), we get

$$\mathbf{E} \cdot \mathbf{F}_{\partial W} = -\iiint_W \text{ div } (p\mathbf{E})\, dx\, dy\, dz = -\iiint_W \text{ (grad } p) \cdot \mathbf{E}\, dx\, dy\, dz,$$

so that

$$\mathbf{F}_{\partial W} = -\iiint_W \nabla p\, dx\, dy\, dz.$$

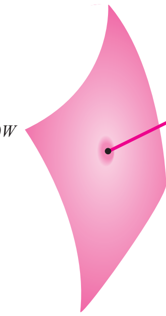


FIGURE 8.5.2. The force acting on ∂W per unit area is $-p\mathbf{n}$.

Now we apply *Newton's second law* to a moving region W_t. As in the transport theorem, $W_t = \phi_t(W)$, where $\phi_t(\mathbf{x}) = \phi(\mathbf{x}, t)$ denotes the flow of $\mathbf{V}$. The rate of change of momentum of the fluid in W_t equals the force acting on it:

$$\frac{d}{dt}\iiint_{W_t} p\mathbf{V}\,dx\,dy\,dz = \mathbf{F}_{\partial W_t} = \iiint_{W_t} \nabla p\,dx\,dy\,dz.$$

We apply the vector form of the transport theorem to the left-hand side to get

$$\iiint_{W_t}\left[\frac{\partial}{\partial t}(\rho\mathbf{V}) + \mathbf{V}\cdot\nabla(\rho\mathbf{V}) + p\mathbf{V}\ \mathrm{div}\ \mathbf{V}\right]dx\,dy\,dz = -\iiint_{W_t}\nabla p\,dx\,dy\,dz.$$

Because W_t is arbitrary, this is equivalent to

$$\frac{\partial}{\partial t}(\rho\mathbf{V}) + \mathbf{V}\cdot\nabla(\rho\mathbf{V}) + \rho\mathbf{V}\ \mathrm{div}\ \mathbf{V} = -\nabla p.$$

Simplification using the equation of continuity, namely, formula $(1')$, gives

$$\rho\left(\frac{\partial\mathbf{V}}{\partial t} + \mathbf{V}\cdot\nabla\mathbf{V}\right) = -\nabla p. \tag{8.5.2}$$

This is *Euler's equation for a perfect fluid.* For compressible fluids, p is a given function of ρ (for instance, for many gases, $p = A\rho^\gamma$ for constants A and γ). On the other hand, if the fluid is incompressible, ρ is to be determined from the condition div $\mathbf{V} = 0$. Equations (1) and (2) then govern the motion of the fluid.

The equations describing the motion of a fluid were first derived by Leonhard Euler in 1755, in a paper entitled "General Principles of the Motion of Fluids." Euler did basic work in mechanics as well as voluminous work in pure mathematics, a small part of which has already been discussed in this book; he essentially began the subject of analytical mechanics (as opposed to the Euclidean geometric methods used by Newton). He is responsible for the equations of a rigid body (equations that apply, for example, to a tumbling satellite) and the formulation of many basic equations of mechanics in terms of variational principles; that is, by the methods of maxima and minima of real-valued functions. Euler wrote the first comprehensive textbook on calculus and contributed to virtually all branches of mathematics. He wrote several books and hundreds of research papers even after he became totally blind, and he was working on a new treatise on fluid mechanics at the time of his death in 1783. Euler's equations for a fluid were eventually modified by Navier and Stokes to include viscous effects; the resulting Navier–Stokes equations are described in virtually every textbook on fluid mechanics.[7] Stokes is, of course, also responsible for developing Stokes' theorem, one of the main results discussed in this text!

Conservation of Energy and the Derivation of the Heat Equation

If $T(t, x, y, z)$ (a C^2 function) denotes the temperature in a body at time t, then ∇T represents the temperature gradient: Heat "flows" with the vector field $-\nabla T = \mathbf{F}$. Note that ∇T points in the direction of *increasing* T. Because heat flows from hot to cold, we have inserted a minus sign to reflect this physically observable fact. The energy density, that is, the energy per unit volume, is $c\rho_0 T$, where c is a constant (called the specific heat) and ρ_0 is the mass density, assumed constant. (We accept these assertions from elementary physics.) The *energy flux vector* is defined to be $\mathbf{J} = k\mathbf{F}$, where k is a constant called the *conductivity.*

One now makes the hypothesis that energy is conserved. This means that $\mathbf{J}$ and $\rho = c\rho_0 T$ should obey the law of conservation of mass, with ρ playing the role of "mass" (note that it is *energy density*, not mass); that is,

$$\frac{d}{dt}\iiint_W \rho\, dV = -\iint_{\partial W} \mathbf{J}\cdot\mathbf{n}\, dS.$$

By Theorem 11, this assertion is equivalent to

$$\text{div}\ \mathbf{J} + \frac{\partial \rho}{\partial t} = 0.$$

[7] The Clay Foundation has offered a prize of \$1 million to anyone who shows that for the incompressible Navier–Stokes equations, smooth data at $t = 0$ lead to smooth solutions for all $t > 0$.

But

$$\text{div } \mathbf{J} = \text{ div}\,(-k\nabla T) = -k\nabla^2 T.$$

(Recall that $\nabla^2 T = \partial^2 T/\partial x^2 + \partial^2 T/\partial y^2 + \partial^2 T/\partial z^2$ and ∇^2 is the Laplace operator.) Continuing, we have

$$\frac{\partial \rho}{\partial t} = \frac{\partial (c\rho_0 T)}{\partial t} = c\rho_0 \frac{\partial T}{\partial t}.$$

Thus, the equation div $\mathbf{J} + \partial\rho/\partial t = 0$ becomes

$$\frac{\partial T}{\partial t} = \frac{k}{c\rho_0}\nabla^2 T = k\nabla^2 T, \tag{3}$$

where $\kappa = k/c\rho_0$ is called the *diffusivity.* Equation (3) is the important *heat equation.*

Just as equations (1) and (2) govern the flow of an ideal fluid, equation (3) governs the conduction of heat in the following sense. If $T(0,\, x,\, y,\, z)$ is a given initial temperature distribution, then a unique $T(t,\, x,\, y,\, z)$ is determined that satisfies equation (3). In other words, the initial condition at $t = 0$ gives the result for t ¿ 0. Notice that if T does not change with time (the steady-state case), then we must have $\nabla^2 T = 0$ (Laplace's equation).

Next we show how vector analysis can be used to solve differential equations by a method called ***potential theory*** or the ***Green's-function method.*** The presentation will be quite informal; the reader may consult references, such as

G.F.D. Duff and D. Naylor, *Differential Equations of Applied Mathematics*, Wiley, New York, 1966,

R. Courant and D. Hilbert, *Methods of Mathematical Physics*. Volumes I and II, John Wiley & Sons Inc., New York, 1989. Wiley Classics Library, Reprint of the 1962 original, A Wiley-Interscience Publication.

for further information.

Suppose we wish to solve Poisson's equation

$$\nabla^2 u = \rho$$

for $u(x, y, z)$, where $\rho(x, y, z)$ is a given function. Recall that this equation arises from Gauss' Law if $\mathbf{E} = \nabla u$ and also in the problem of determining the gravitational potential from a given mass distribution.

A function $G(\mathbf{x}, \mathbf{y})$ that has the properties

$$G(\mathbf{x}, \mathbf{y}) = G(\mathbf{y}, \mathbf{x}) \quad \text{and} \quad \nabla^2 G(\mathbf{x}, \mathbf{y}) = \delta(\mathbf{x} - \mathbf{y}) \tag{4}$$

(in this expression $\mathbf{y}$ is *held fixed*, that is, which solves the differential equations with ρ replaced by δ, is called ***Green's function*** for this differential equation. Here $\rho(\mathbf{x}-\mathbf{y})$ represents the Dirac delta function, "defined" by[8]

(i) $\delta(\mathbf{x}-\mathbf{y}) = 0 \quad \text{for} \quad \mathbf{x} \neq \mathbf{y}$

and

(ii) $\iiint_{\mathbb{R}^3} \delta(\mathbf{x}-\mathbf{y})\, d\mathbf{y} = 1.$

It has the following operational property that formally follows from conditions (i) and (ii): For any continuous function $f(x)$,

$$\iiint_{\mathbb{R}^3} f(\mathbf{y})\delta(\mathbf{x}-\mathbf{y})\, d\mathbf{y} = f(\mathbf{x}). \tag{5}$$

This is sometimes called the ***sifting property*** of δ.

Theorem 1. *If $G(\mathbf{x},\mathbf{y})$ satisfies the differential equation $\nabla^2 u = \rho$ with ρ replaced by $\delta(\mathbf{x}-\mathbf{y})$, then*

$$u(\mathbf{x}) = \iiint_{\mathbb{R}^3} G(\mathbf{x},\mathbf{y})\rho(\mathbf{y})\, d\mathbf{y} \tag{6}$$

is a solution to $\nabla^2 u = \rho$.

Proof. To see this, note that

$$\begin{aligned}
\nabla^2 \iiint_{\mathbb{R}^3} G(\mathbf{x},\mathbf{y})\rho(\mathbf{y})\, d\mathbf{y} &= \iiint_{\mathbb{R}^3} (\nabla^2 G(\mathbf{x},\mathbf{y})\rho(\mathbf{y})\, d\mathbf{y} \\
&= \iiint_{\mathbb{R}^3} \delta(\mathbf{x}-\mathbf{y})\rho(\mathbf{y})\, d\mathbf{y} && \text{(by 4)} \\
&= \rho(\mathbf{x}) && \text{(by 5)}
\end{aligned}$$

■

The "function" $\rho(\mathbf{x}) = \delta(\mathbf{x})$ represents a unit change concentrated at a single point [see conditions (i) and (ii), above]. Thus $G(\mathbf{x},\mathbf{y})$ *represents the potential at* $\mathbf{x}$ *due to a charge placed at* $\mathbf{y}$.

Green's Function in $\mathbb{R}^3$. We claim that equation (4) is satisfied if we choose

$$G(\mathbf{x},\mathbf{y}) = -\frac{1}{4\pi\|\mathbf{x}-\mathbf{y})\|}.$$

Clearly $G(\mathbf{x},\mathbf{y}) = G(\mathbf{y},\mathbf{x})$. To check the second part of equation (4), we must verify that $\nabla^2 G(\mathbf{x},\mathbf{y})$ has the following two properties of the δ function:

[8]This is not a precise definition; nevertheless, it is enough here to assume that δ is a symbolic expression with the operational property shown in equation (5). See the references in the preceding footnote for a more careful definition of δ.

(i) $\nabla^2 G(\mathbf{x},\mathbf{y}) = 0 \quad$ for $\mathbf{x} \neq \mathbf{y}$

and

(ii) $\iiint_{\mathbb{R}^3} \nabla^2 G(\mathbf{x},\mathbf{y}) = 1$.

We will explain the meaning of (ii) in the course of the following discussion. Property (i) is true because the gradient of G is

$$\nabla G(\mathbf{x},\mathbf{y}) = \frac{\mathbf{r}}{4\pi r^3}, \tag{7}$$

where $\mathbf{r} = \mathbf{x} - \mathbf{y}$ is the vector from $\mathbf{y}$ to $\mathbf{x}$ and $r = \|\mathbf{r}\|$ (see Exercise 30, §4.4), and therefore for $r \neq 0, \nabla \cdot \nabla G(\mathbf{x},\mathbf{y}) = 0$ (as in the aforementioned exercise). For property (ii), let B be a ball about $\mathbf{x}$; by property (i),

$$\iiint_{\mathbb{R}^3} \nabla^2 G(\mathbf{x},\mathbf{y})\ d\mathbf{y} = \iiint_B \nabla^2 G(\mathbf{x},\mathbf{y})\ d\mathbf{y}.$$

This, in turn, equals

$$\iint_{\partial B} \nabla^2 G(\mathbf{x},\mathbf{y}) \cdot \mathbf{n}\ dS.$$

by Gauss' Theorem. Thus, making use of (7), we get

$$\iint_{\partial B} \nabla^2 G(\mathbf{x},\mathbf{y}) \cdot \mathbf{n}\ dS = \iint_{\partial B} \frac{\mathbf{r}\cdot\mathbf{n}}{4\pi r^3}\ dS = 1,$$

which proves property (ii). Therefore, *a solution of* $\nabla^2 u = \rho$ *is*

$$u(\mathbf{x}) = \iiint_{\mathbb{R}^3} \frac{-\rho(\mathbf{y})}{4\pi\|\mathbf{x}-\mathbf{y}\|}\ d\mathbf{y}. \tag{8}$$

by Theorem 1.

Green's Function in Two Dimensions. In the plane rather than in $\mathbb{R}^3$, one can similarly show that

$$G(\mathbf{x},\mathbf{y}) = \frac{1}{2\pi}\log\|\mathbf{x}-\mathbf{y}\|, \tag{9}$$

and so a solution of the equation $\nabla^2 u = \rho$ is

$$u(\mathbf{x}) = \frac{1}{2\pi}\iint_{\mathbb{R}^2} \rho(\mathbf{y})\log\|\mathbf{x}-\mathbf{y}\|\ d\mathbf{y}.$$

Green's Identities. We now turn to the problem of using Green's functions to solve Poisson's equation in a bounded region with given boundary conditions. To do this, we need Green's first and second identities, which can be obtained from the divergence theorem (see Exercise 15, §8.4). We start with the identity

$$\iiint_V \nabla \cdot \mathbf{F}\ dV = \iint_S \mathbf{F}\cdot\mathbf{n}\ dS.$$

where V is a region in space, S is its boundary, and $\mathbf{n}$ is the outward unit normal vector. Replacing $\mathbf{F}$ by $f\nabla g$, where f and g are scalar functions, we obtain

$$\iiint_V \nabla f \cdot \nabla g \; dV + \iiint_V f\nabla^2 g \; dV = \iint_S f\frac{\partial g}{\partial n} \; dS. \tag{10}$$

where $\partial g/\partial n = \nabla g \cdot \mathbf{n}$. This is ***Green's first identity.*** If we simply permute f and g and subtract the result from equation (10), we obtain ***Green's second identity***,

$$\iiint_V (f\nabla^2 g - g\nabla^2 f) \; dV = \iint_S \left(f\frac{\partial g}{\partial n} - \frac{\partial f}{\partial n}\right) \; dS, \tag{11}$$

which we shall use shortly.

Green's Functions in Bounded Regions. Consider Poisson's equation $\nabla^2 u = \rho$ in some region V, and the corresponding equations for Green's function

$$G(\mathbf{x},\mathbf{y}) = G(\mathbf{y},\mathbf{x}) \quad \text{and} \quad \nabla^2 G(\mathbf{x},\mathbf{y}) = \delta(\mathbf{x}-\mathbf{y}).$$

Inserting u and G into Green's second identity (11), we obtain

$$\iiint_V (u\nabla^2 G - G\nabla^2 u) \; dV = \iint_S \left(u\frac{\partial G}{\partial n} - G\frac{\partial u}{\partial n}\right) dS.$$

Choosing our integration variable to be $\mathbf{y}$ and using $G(\mathbf{x},\mathbf{y}) = G(\mathbf{y},\mathbf{x})$, this becomes

$$\iiint_V [u(\mathbf{y})\delta(\mathbf{x}-\mathbf{y}) - G(\mathbf{x},\mathbf{y})\rho(\mathbf{y})] \; d\mathbf{y} = \iint_S \left(u\frac{\partial G}{\partial n} - G\frac{\partial u}{\partial n}\right) dS;$$

and by equation (5),

$$u(\mathbf{x}) = \iiint_V G(\mathbf{x},\mathbf{y})\rho(\mathbf{y}) \; d\mathbf{y} + \iint_S \left(u\frac{\partial G}{\partial n} - G\frac{\partial u}{\partial n}\right) dS. \tag{12}$$

Note that for an unbounded region, this becomes identical to our previous result, equation (6), for all of space. Equation (12) enables us to solve for u in a bounded region where $\rho = 0$ by incorporating the conditions that u must obey on S.

If $\rho = 0$, equation (12) reduces to

$$u = \iint_S \left(u\frac{\partial G}{\partial n} - G\frac{\partial u}{\partial n}\right) dS,$$

or, fully,

$$u(\mathbf{x}) = \iint_S \left[u(\mathbf{y})\frac{\partial G}{\partial n}(\mathbf{x},\mathbf{y}) - G(\mathbf{x},\mathbf{y})\frac{\partial u}{\partial n}(\mathbf{y})\right] \; dS, \tag{13}$$

where u appears on both sides of the equation and the integration variable is $\mathbf{y}$. The crucial point is that evaluation of the integral requires only that we know the behavior of u on S. Commonly, either u is given on the boundary (for a ***Dirichlet problem***) or $\partial u/\partial n$ is given on the boundary (for a ***Neumann problem***. If we know u on the boundary, we want to make $G\partial u/\partial n$ vanish on the boundary so we can evaluate the integral. Therefore if u is given on S we must find a G such that $G(\mathbf{x}, \mathbf{y})$ vanishes whenever $\mathbf{y}$ lies on S. This is called the ***Dirichlet Green's function for the region*** V. Conversely, if $\partial u/\partial n$ is given on S we must find a G such that $\partial G/\partial n$ vanishes on S. This is the ***Neumann Green's function.***

Thus, a Dirichlet Green's function $G(\mathbf{x}, \mathbf{y})$ is defined for $\mathbf{x}$ and $\mathbf{y}$ in the volume V and satisfies these three conditions:

(a) $G(\mathbf{x}, \mathbf{y}) = G(\mathbf{y}, \mathbf{x})$,

(b) $\nabla^2 G(\mathbf{x}, \mathbf{y}) = \delta(\mathbf{x} - \mathbf{y})$,

(c) $G(\mathbf{x}, \mathbf{y}) = 0$ when $\mathbf{y}$ lies on S, the boundary of the region V.

[Note that by condition (a), in conditions (b) and (c) the variables $\mathbf{x}$ and $\mathbf{y}$ can be interchanged without changing the condition.]

It is interesting to note that *condition* (a) *is actually a consequence of conditions* (b) *and* (c), *provided* (b) *and* (c) *also hold with* $\mathbf{x}$ *and* $\mathbf{y}$ *interchanged.*

To see this, we *fix points* $\mathbf{y}$ and $\mathbf{w}$ and use equation (11) with $f(\mathbf{x}) = G(\mathbf{x}, \mathbf{y})$ and $g(\mathbf{x}) = G(\mathbf{w}, \mathbf{x})$. By condition (b),

$$\nabla^2 f(\mathbf{x}) = \delta(\mathbf{x} - \mathbf{y}) \quad \text{and} \quad \nabla^2 g(\mathbf{x}) = \delta(\mathbf{x} - \mathbf{w}),$$

and by condition (c), f and g vanish when $\mathbf{x}$ lies on S and so the right hand side of (11) is zero. On the left hand side, we substitute $f(\mathbf{x})$ and $g(\mathbf{x})$ to give

$$\iiint_V [G(\mathbf{x}, \mathbf{y})\delta(\mathbf{x} - \mathbf{w}) - G(\mathbf{w}, \mathbf{x})\delta(\mathbf{x} - \mathbf{y})]\; d\mathbf{x} = 0$$

which gives

$$G(\mathbf{w}, \mathbf{y}) = G(\mathbf{y}, \mathbf{w}),$$

which is the asserted symmetry of G. This means, in effect, that if (b) and (c) hold with $\mathbf{x}$ and $\mathbf{y}$ internchanged, then it is not necessary to check condition (a). (This result is sometimes called the ***principle of reciprocity.***)

Solving any particular Dirichlet or Neumann problem thus reduces to the task of finding Laplace's equations on all of $\mathbb{R}^2$ or $\mathbb{R}^3$, namely, equations (8) and (9).

Green's Function for a Disk. We shall now use the two-dimensional Green's function method to construct the Dirichlet Green's function for the disk of radius R (see Figure 8.5.3). This will enable us to solve $\nabla^2 u = 0$ (or $\nabla^2 u = \rho$) with u given on the boundary circle.

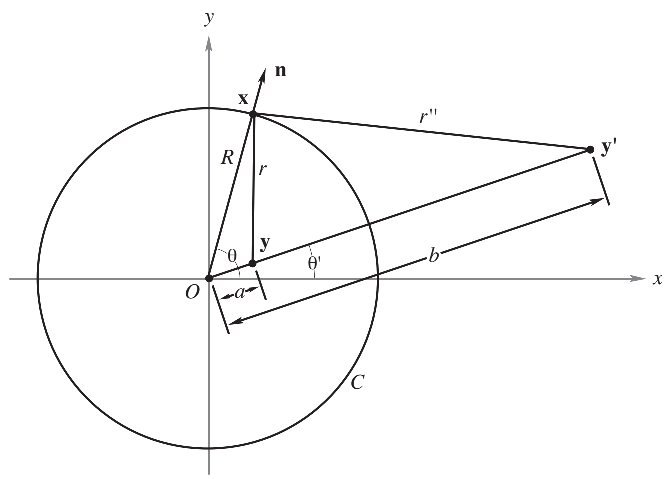


FIGURE 8.5.3. Geometry of the construction of Green's function for a disk.

In Figure 8.5.3 we have draw the point $\mathbf{x}$ on the circumference because that is where we want G to vanish. [According to the procedure above, $G(\mathbf{x},\mathbf{y})$ is supposed to vanish when either $\mathbf{x}$ or $\mathbf{y}$ is on C. We have chosen $\mathbf{x}$ on C to begin with.] The Green's function $G(\mathbf{x},\mathbf{y})$ that we shall find will, of course, be valid for all $\mathbf{x},\mathbf{y}$ in the disk. The point $\mathbf{y}'$ represents the "reflection" of the point $\mathbf{y}$ into the region outside the circle, such that $ab = R^2$. Now when $\mathbf{x} \in C$, by the similarity of the triangles $\mathbf{x}O\mathbf{y}$ and $\mathbf{x}O\mathbf{y}'$,

$$\frac{r}{R} = \frac{r''}{b}, \quad \text{that is,} \quad r = \frac{r''R}{b} = \frac{r''a}{R}.$$

Hence, if we choose our Green's function to be

$$G(\mathbf{x},\mathbf{y}) = \frac{1}{2\pi}\left(\log r - \log\frac{r''a}{R}\right), \tag{14}$$

we see that G is zero if $\mathbf{x}$ is on C. Since $r''a/R$ reduces to r when $\mathbf{y}$ is on C, G also vanishes when $\mathbf{y}$ is on C. If we can show that G satisfies $\nabla^2 G = \delta(\mathbf{x}-\mathbf{y})$ in the circle, then we will have proved that G is indeed the Dirichlet Green's function. From equation (9) we know that $\nabla^2(logr)/2\pi = \delta(\mathbf{x}-\mathbf{y})$, so that

$$\nabla^2 G(\mathbf{x},\mathbf{y}) = \delta(\mathbf{x}-\mathbf{y}) - \delta(\mathbf{x}-\mathbf{y}'),$$

but $\mathbf{y}'$ is always outside the circle, and so $\mathbf{x}$ can never be equal to $\mathbf{y}'$ and $\delta(\mathbf{x}-\mathbf{y}')$ is always zero. Hence,

$$\nabla^2 G(\mathbf{x},\mathbf{y}) = \delta(\mathbf{x}-\mathbf{y})$$

and thus G is the Dirichlet Green's function for the circle.

Now we shall consider the problem of solving

$$\nabla^2 u = 0$$

in this circle if $u(R, \theta) = f(\theta)$ is the given boundary condition. By equation (13) we have a solution

$$u = \int_C \left(u \frac{\partial G}{\partial n} - G \frac{\partial u}{\partial n} \right) \, ds.$$

But $G = 0$ on C, and so we are left with the integral

$$u = \int_C u \frac{\partial G}{\partial n} \, ds,$$

where we can replace u by $f(\theta)$, since the integral is around C. Thus the task of solving the Dirichlet problem in the circle is reduced to finding $\partial G/\partial n$. From equation (14) we can write

$$\frac{\partial G}{\partial n} = \frac{1}{2\pi} \left(\frac{1}{r} \frac{\partial r}{\partial n} - \frac{1}{r''} \frac{r''}{\partial n} \right).$$

Now

$$\frac{\partial r}{\partial n} = \nabla r \cdot \mathbf{n} \quad \text{and} \quad \nabla r = \frac{\mathbf{r}}{r},$$

where $\mathbf{r} = \mathbf{x} - \mathbf{y}$, and so

$$\frac{\partial r}{\partial n} = \frac{\mathbf{r} \cdot \mathbf{n}}{r} = \frac{r \cos(nr)}{r} = \cos(nr),$$

where (nr) represents the angle between $\mathbf{n}$ and $\mathbf{r}$. Likewise,

$$\frac{\partial r''}{\partial n} = \cos(nr'').$$

In triangle $\mathbf{xy}O$, we have, by the cosine law, $a^2 = r^2 + R^2 - 2rR\cos(nr)$, and in triangle $\mathbf{xy'0}$, we get $b^2 = (r'')^2 + R^2 - 2r''R\cos(nr'')$, and so

$$\frac{\partial r}{\partial n} = \cos(nr) = \frac{R^2 + r^2 - a^2}{2rR} \quad \text{and} \quad \frac{\partial r''}{\partial n} = \cos(nr'') = \frac{R^2 + (r'')^2 - a^2}{2r''R}.$$

Hence

$$\frac{\partial G}{\partial n} = \frac{1}{2\pi} \left[\frac{R^2 + r^2 - a^2}{2rR} - \frac{R^2 + (r'')^2 - a^2}{2r''R} \right].$$

Using the relationship between r and r'' when $\mathbf{x}$ is on C, we get

$$\left. \frac{\partial G}{\partial n} \right|_{\mathbf{x} \in C} = \frac{1}{2\pi} \left(\frac{R^2 - a^2}{Rr^2} \right).$$

Thus the solution can be written as

$$u = \frac{1}{2\pi} \int_C f(\theta) \frac{R^2 - a^2}{Rr^2}\ ds.$$

Let us write this in a more useful form. Note that in triangle $\mathbf{xy0}$, we can write

$$r = [a^2 + R^2 - 2aR\cos(\theta - \theta')]^{1/2},$$

where θ and θ' are the polar angles in $\mathbf{x}$ and $\mathbf{y}$ space, respectively. Second, our solution must be valid for all $\mathbf{y}$ in the circle; hence the distance of $\mathbf{y}$ from the origin must now become a variable, which we shall call r'. Finally, note that $ds = R\ d\theta$ on C, so we can write the solution in polar coordinates as

$$u(r', \theta') = \frac{R^2 - (r')^2}{2\pi} \int_0^{2\pi} \frac{f(\theta)\ d\theta}{(r')^2 + R^2 - 2r'R\cos(\theta - \theta')}.$$

This is know as ***Poisson's formula in two dimensions.***[9] As an exercise, the reader should use this to write down the solution of $\nabla^2 u = \rho$ with u a given function $f(\theta)$ on the boundary.

Exercises

1. Use a direct argument (or the proof of Theorem 1 in the Internet supplement to Section 4.4) to show that

$$\frac{\partial}{\partial t} J(\mathbf{x}, t) = [\text{div}\ \mathbf{F}(\phi(\mathbf{x}, t))] J(\mathbf{x}, t).$$

[9] There are several other ways of deriving this famous formula. For the method of complex variables, see J. Marsden and M. Hoffman, *Basic Complex Analysis*, 2d ed., Freeman, New York, 1987, p. 195. For the method of Fourier series, see J. Marsden, *Elementary Classical Analysis*, Freeman, New York, 1974, p. 466.

2. Using the change of variables theorem and Exercise 1, show that if $f(x, y, z, t)$ is a given function and $W \subset \mathbb{R}^3$ is any region, then the *transport equation* holds:

$$\frac{d}{dt} \iiint_{W_t} f(x, y, z, t)\, dx\, dy\, dz = \iiint_{W_t} \left(\frac{Df}{Dt} + f \text{div } \mathbf{F} \right) dx\, dy\, dz$$

where $W_t = \phi_t(W)$, which is the region moving with the flow, and where $Df/Dt = \partial f/\partial t + \nabla f \cdot \mathbf{F}$ is the material derivative.

3. Use the transport equation to show that

$$\frac{d}{dt} \iiint_{W_t} \rho\, dx\, dy\, dz = 0$$

is equivalent to the law of conservation of mass.

4. Using Exercise 3 and the change of variables theorem, show that $\rho(\mathbf{x}, t)$ can be expressed in terms of the Jacobian $J(\mathbf{x}, t)$ of the flow map $\phi(\mathbf{x}, t)$ and $\rho(\mathbf{x}, 0)$ by the equation

$$\rho(\mathbf{x}, t) J(\mathbf{x}, t) = \rho(\mathbf{x}, 0).$$

What can you conclude from this for incompressible flow?

5. Prove the vector form of the transport theorem, namely,

$$\frac{d}{dt} \iiint_{W_t} (f\mathbf{F})\, dx\, dy\, dz = \iiint_{W_t} \left[\frac{\partial}{\partial t}(f\mathbf{F}) + \mathbf{F} \cdot \nabla(f\mathbf{F}) + (f\mathbf{F}) \text{ div } \mathbf{F} \right] dx\, dy\, dz,$$

where $\mathbf{F} \cdot \nabla(f\mathbf{F})$ denotes the 3×3 derivative matrix $\mathbf{D}\,(f\mathbf{F})$ operating on the column vector $\mathbf{F}$; in Cartesian coordinates, $\mathbf{F} \cdot \nabla \mathbf{G}$ is the vector whose ith component is

$$\sum_{j=1}^{3} F_j \frac{\partial G^i}{\partial x_j} = F_1 \frac{\partial G^i}{\partial x} + F_2 \frac{\partial G^i}{\partial y} + F_3 \frac{\partial G^i}{\partial z}.$$

6. Let $\mathbf{V}$ be a vector field with flow $\phi(\mathbf{x}, t)$ and let $\mathbf{V}$ and ρ satisfy the law of conservation of mass. Let W_t be the region transported with the flow. Prove the following version of the transport theorem:

$$\frac{d}{dt} \iiint_{W_t} \rho f\, dx\, dy\, dz = \iiint_{W_t} \rho \frac{Df}{Dt}\, dx\, dy\, dz.$$

7. (*Bernoulli's law*) (a) Let $\mathbf{V}$, ρ satisfy the law of conservation of mass and equation (2) (Euler's equation for a perfect fluid). Suppose $\mathbf{V}$ is

irrotational and hence that $\mathbf{V} = \nabla\phi$ for a function ϕ. Show that if C is a path connecting two points P_1 and P_2, then

$$\left(\frac{\partial \phi}{\partial t} + \frac{1}{2}\|\mathbf{V}\|^2\right)\Big|_{P_1}^{P_2} + \int_C \frac{dp}{\rho} = 0.$$

[Hint: You may need the vector identity, $(\mathbf{V}\cdot\nabla)\mathbf{V} = \frac{1}{2}\nabla(\|\mathbf{V}\|^2) + (\nabla\times\mathbf{V})\times\mathbf{V}$.]

(b) If in part (a), $\mathbf{V}$ is stationary—that is, $\partial\mathbf{V}/\partial t = 0$—and ρ is constant, show that

$$\frac{1}{2}\|\mathbf{V}\|^2 + \frac{p}{\rho}$$

is constant in space. Deduce that, in this situation, *higher pressure is associated with lower fluid speed.*

8. Using Exercise 7, show that if ϕ satisfies Laplace's equation $\nabla^2\phi = 0$, then $\mathbf{V} = \nabla\phi$ is a stationary solution to Euler's equation for a perfect *incompressible* fluid with constant density.

9. Verify that Maxwell's equations imply the equation of continuity for $\mathbf{J}$ and ρ.

10. (a) With notation as in Figure 8.5.3, show that the Dirichlet problem for the sphere of radius R in three dimensions has Green's function

$$G(\mathbf{x},\mathbf{y}) = \frac{1}{4\pi}\left(\frac{R}{ar''} - \frac{1}{r}\right).$$

(b) Prove Poisson's formula in three dimensions:

$$u(\mathbf{y}) = \frac{R(R^2-a^2)}{4\pi}\int_0^{2\pi}\int_0^{\pi}\frac{f(\theta,\phi)\sin\phi\, d\theta d\phi}{(R^2-a^2-2Ra\cos\gamma)^{3/2}}$$

11. Let H denote the upper half space $z \geq 0$. For a point $\mathbf{x} = (x,y,z)$ in H, let $R(\mathbf{x}) = (x,y,-z)$, the reflection of $\mathbf{x}$ in the xy plane. Let $G(\mathbf{x},\mathbf{y}) = -1/(4\pi\|\mathbf{x}-\mathbf{y}\|)$ be the Green's function for all of $\mathbb{R}^3$.

(a) Verify that the function $\tilde{G}$ defined by

$$\tilde{G}(\mathbf{x},\mathbf{y}) = G(\mathbf{x},\mathbf{y}) - G(R(\mathbf{x}),\mathbf{y})$$

is the Green's function for the Laplacian in H.

(b) Write down a formula for the solution u of the problem

$$\nabla^2 u = \rho \quad \text{in} \quad H, \qquad \text{and} \qquad u(x,y,0) = \phi(x,y).$$

Exercises 3 through 9 give some sample applications of vector calculus to shock waves.[10]

12. Consider the equation

$$u_t + uu_x = 0$$

for a function $u(x,t), -\infty < x < \infty, t \geq 0$, where $u_t = \partial u/\partial t$ and $u_x = \partial u/\partial x$. Let $u(x,0) = u_0(x)$ be the given value of u at $t = 0$. The curves $(x(s), t(s))$ in the xt plane defined by

$$\dot{x} = u, \dot{t} = 1$$

are called ***characteristic curves*** (the overdot denotes the derivative with respect to s).

(a) Show that u is constant along each characteristic curve by showing that $\dot{u} = 0$.

(b) Show that the slopes of the characteristic curves are given by $dt/dx = 1/u$, and use it to prove that the characteristic curves are straight lines determined by the initial data.

(c) Suppose that $x_1 < x_2$ and $u_0(x_1) > u_0(x_2) > 0$. Show that the two characteristics through the points $(x_1, 0)$ and $(x_2, 0)$ intersect at a point $P = (\bar{x}, \bar{t})$ with $\bar{t} > 0$. Show that this together with the result in part (a) implies that the solution cannot be continuous at P (see Figure 8.5.4).

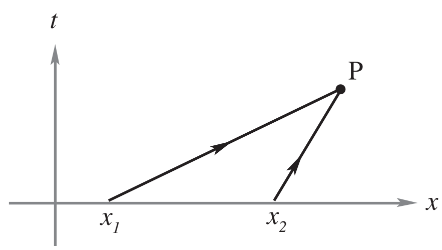


FIGURE 8.5.4. Characteristics of the equation $u_t + uu_x = 0$.

(d) Calculate $\bar{t}$.

13. Repeat Exercise 3 for the equation

$$u_t + f(u)_x = 0, \qquad (15)$$

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[10] For additional details, consult A. J. Chorin and J. E. Marsden, *A Mathematical Introduction to Fluid Mechanics,* 3rd ed., Springer-Verlag, New York, 1992, and P. D. Lax, "The Formation and Decay of Shock Waves," *Am. Math. Monthly* 79 (1972): 227-241. We are grateful to Joel Smoller for suggesting this series of exercises.

where $f'' > 0$ and $f'(u_0(x_2)) > 0$. The characteristics are now defined by $\dot{x} = f'(u), \dot{t} = 1$. We call equation (15) and equation in ***divergence form.*** (This exercise shows that a continuous solution is generally impossible—irrespective of the smoothness of f!)

14. *(Weak solutions)* Since equations of the form in Exercise 4 arise in many physical applications [gas dynamics, magnetohydrodynamics, nonlinear optics (lasers)] and because it would be nice for a solution to exist for all time (t), it is desirable to make sense out of the equation by reinterpreting it when discontinuities develop. To this end, let $\phi = \phi(x,t)$ be a C^1 function. Let D be a rectangle in the xt plane determined by $-a \le x \le a$ and $0 \le t \le T$, such that $\phi(x,t) = 0$ for $x = \pm a, x = T$, and for all (x,t) in the upper half plane outside D. Let u be a "genuine" solution of equation (15).

   (a) Show that

$$\iint_{t\ge 0} [u\phi_t + f(u)\phi_x]\ dxdt + \int_{t=0} u_0(x)\phi(x,0)\ dx = 0. \quad (16)$$

   (HINT: Start with $\iint_D [u_t + f(u)_x]\phi\ dxdx = 0$.)

   Thus, if u is a smooth solution, then equation (16) holds for all ϕ as above. We call the function u a ***weak solution*** of equation (15) if equation (16) holds for all such ϕ.

   (b) Show that if u is a weak solution that is C^1 in an open set Ω in the upper half of the xt plane, then u is a genuine solution of equation (15) in Ω.

15. (*The* ***jump condition***, which is also known in gas dynamics as the ***Rankine-Hugoniot condition***.) The definition of a weak solution given in Exercise 5 clearly allows discontinuous solutions. However, the reader shall now determine that not every type of discontinuity is admissible, for there is a connection between the discontinuity curve and the values of the solution on both sides of the discontinuity.

   Let u be a (weak) solution of equation (15) and suppose Γ is a smooth curve in the xt plane such that u "jumps" across a curve Γ; that is, u is of a class C^1 except for jump discontinuity across Γ. We call that Γ a ***shock wave***. Choose a point $P \in \Gamma$ and construct, near P, a "rectangle" $D = D_1 \cup D_2$, as shown in Figure 8.5.5. Choose ϕ to vanish on D and outside D.

   (a) Show that

$$\iint_D [u\phi_t + f(u)\phi_x]\ dxdt = 0$$

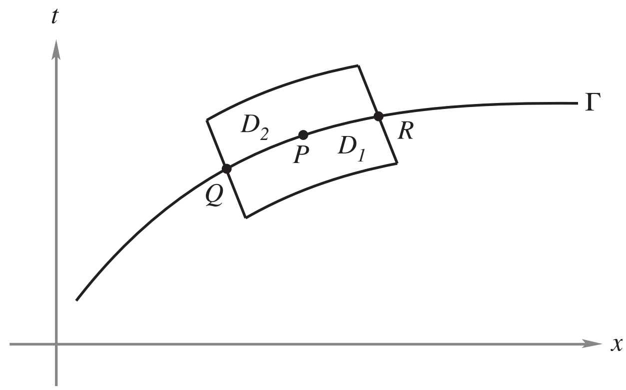

FIGURE 8.5.5. The solution u jumps in value from u_1 to u_2 across Γ.

and

$$\iint_{D_1} [u\phi_t + f(u)\phi_x]\ dxdt = \iint_{D_1} [(u\phi)_t + (f(u)\phi)_x]\ dxdt.$$

(b) Suppose that u jumps in value from u_1 to u_2 across Γ so that when (x, t) approaches a point (x_0, t_0) on Γ from $\partial D_i, u(x, t)$ approaches the value $u_i(x_0, t_0)$. Show that

$$0 = \int_{\partial D_1} \phi[-u\ dx + f(u)\ dt] + \int_{\partial D_2} \phi[-u\ dx + f(u)\ dt]$$

and deduce that

$$0 = \int_{\Gamma} \phi([-u]\ dx + [f(u)]\ dt)$$

where $[\alpha(u)] = \alpha(u_2) - \alpha(u_1)$ denotes the jump in the quantity $\alpha(u)$ across Γ.

(c) If the curve Γ defines x implicitly as a function of t and ∂D intersects Γ at $Q = (x(t_1), t_1)$ and $R = (x(t_2), t_2)$, show that

$$0 = \int_Q^R \phi([-u]\ dx + [f(u)]\ dt) = \int_{t_1}^{t_2} \phi\left([-u]\frac{dx}{dt} + [f(u)]\right) dt.$$

(d) Show that at the point P on Γ,

$$[u] \cdot s = [f(u)], \tag{17}$$

where $s = dx/dt$ at P. The number s is called the ***speed*** of the discontinuity. Equation (17) is called the ***jump condition***; it is the relationship that any discontinuous solution will satisfy.

16. *(Loss of uniqueness)* One drawback of accepting weak solutions is loss of uniqueness. (In gas dynamics, some mathematical solutions are extraneous and rejected on physical grounds. For example, discontinuous solutions of rarefaction shock waves are rejected because they indicate that entropy *decreases* across the discontinuity.)

    Consider the equation

$$u_t + \left(\frac{u^2}{2}\right)_x = 0, \quad \text{with initial data} \quad u(x,0) = \begin{cases} -1, & x \geq 0 \\ 1, & x < 0. \end{cases}$$

    Show that for every $\alpha \geq 1, u_\alpha$ is a weak solution, where u_α is defined by

$$u_\alpha(x,t) = \begin{cases} 1, & x \leq \dfrac{1-\alpha}{2}t \\ -\alpha, & \dfrac{1-\alpha}{2}t \leq x \leq 0 \\ \alpha, & 0 \leq x \leq \dfrac{\alpha-1}{2}t \\ -1, & \dfrac{\alpha-1}{2}t < x. \end{cases}$$

    (It can be shown that if $f'' > 0$, uniqueness can be recovered by imposing an additional constraint on the solutions. Thus, there is a unique solution satisfying the "entropy" condition

$$\frac{u(x+a,t) - u(x,t)}{a} \leq \frac{E}{t}$$

    for some $E > 0$ and all $a \neq 0$. Hence for fixed $t, u(x,t)$ can only "jump down" as x increases. In our example, this holds only for the solution with $\alpha = 1$.)

17. (The solution of equation (15) depends on the particular divergence form used.) The equation $u_t + uu_x = 0$ can be written in the two divergence forms

$$u_t + (\frac{1}{2}u^2)_x = 0. \tag{i}$$

$$(\frac{1}{2}u^2)_t + (\frac{1}{3}u^3)_x = 0. \tag{ii}$$

    Show that a weak solution of equation (i) need not be a weak solution of equation (ii). [HINT: The equations have different jump conditions: In equation (i), $s = \frac{1}{2}(u_2 + u_1)$, while in equation (ii), $s = \frac{2}{3}(u_2^2 + u_1u_2 + u_1^2)/(u_2 + u_1)$.]

18. (Noninvariance of weak solutions under nonlinear transformation) Consider equation (15) where $f'' > 0$.

(a) Show that the transformation $v = f'(u)$ takes this equation into

$$v_t + vv_x = 0. \tag{18}$$

(b) Show that the above transformation *does not* necessarily map discontinuous solutions of equation (15) into discontinuous solutions of equation (18). (HINT: Check the jump conditions; for equation (18), $s[v] = \frac{1}{2}[v^2]$ implies $s[f'(u)] = \frac{1}{2}[f'(u)^2]$; for equation (15), $s[u] = [f(u)]$.)

19. *Requires a knowledge of complex numbers.* Show that Poisson's formula in two dimensions may be written as

$$u(r', \theta') = \frac{1}{2\pi} \int_0^{2\pi} u(re^{i\theta}) \frac{r^2 - |z'|^2}{|re^{i\theta} - z'|^2} d\theta,$$

where $z' = r'e^{i\theta'}$.

Selected Answers and Solutions for Additional Content

§2.7 Some Technical Differentiation Theorems.

1. $\mathbf{D}f(x,y,z) = \begin{bmatrix} e^x & 0 & 0 \\ 0 & -\sin y & 0 \\ 0 & 0 & \cos z \end{bmatrix}$;

   $\mathbf{D}f$ is a diagonal matrix if each component function f_i depends only on x_i.

3. (a) Let $A = B = C = \mathbb{R}$ with $f(x) = 0$ and $g(x) = 0$ if $x \neq 0$ and $g(0) = 1$. Then $w = 0$ and $g(f(x)) - 1$ for all x.

   (b) If $\epsilon > 0$, let δ_1 and δ_2 be small enough that $D_{\delta_1}(\mathbf{y}_0) \subset B$ and $\|g(\mathbf{y}) - \mathbf{w}\| < \epsilon$ whenever $\mathbf{y} \in B$ and $0 < \|\mathbf{y} - \mathbf{y}_0\| < \delta_2$. Since $g(\mathbf{y}_0) = \mathbf{w}$, the $0 < \|\mathbf{y} - \mathbf{y}_0\|$ restriction may be dropped. Let δ be small enough that $\|f(\mathbf{x}) - \mathbf{y}_0\| < \min(\delta_1, \delta_2)$ whenever $\mathbf{x} \in A$ and $0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta$. Then for such $\mathbf{x}$, $\|f(\mathbf{x}) - \mathbf{y}_0\| < \delta_1$, so that $f(\mathbf{x}) \in B$ and $g(f(\mathbf{x}))$ is defined. Also, $\|f(\mathbf{x}) - \mathbf{y}_0\| < \delta_2$, and so $\|g(f(\mathbf{x})) - \mathbf{w}\| < \epsilon$.

5. Let $\mathbf{x} = (x_1, \ldots, x_n)$ and fix an index k. Then

$$f(\mathbf{x}) = a_{kk}x_k^2 + \sum_{i=1\neq k}^{n} a_{kj}x_kx_j + \sum_{j=1\neq k}^{n} a_{kj}x_kx_j + [\text{terms not involving } x_k]$$

and therefore

$$\frac{\partial f}{\partial x_k} = 2a_{kk}x_k + \sum_{i\neq k} a_{ki}x_i + \sum_{i\neq k} a_{ki}x_i = 2\sum_{j=1}^{n} a_{kj}x_j = (2\mathbf{A}\mathbf{x})_k.$$

Since the kth components agree for each $k, \nabla f(\mathbf{x}) = 2\mathbf{A}\mathbf{x}_k$.

7. The matrix $\mathbf{T}$ of partial derivatives is formed by placing $\mathbf{D}g(x_0)$ and $\mathbf{D}h(y_0)$ next to each other in a matrix so that $\mathbf{T}(x_0, y_0)(x - x_0, y - y_0) = \mathbf{D}g(x_0)(x-x_0)+\mathbf{D}h(y_0)(y-y_0)$. Now use the triangle inequality and the fact that $\|(x - x_0, y - y_0)\|$ is larger than $|x - x_0|$ and $|y - y_0|$ to show that $\|f(x, y) - f(x_0, y_0) - \mathbf{T}(x_0, y_0)(x - x_0, y - y_0)\|/\|(x - x_0, y - y_0)\|$ goes to 0.

9. Use the limit theorems and the fact that the function $g(x) = \sqrt{|x|}$ is continuous. (Prove the last statement.)

11. For continuity at $(0, 0)$, use the fact that

$$\left|\frac{xy}{(x^2 + y^2)^{1/2}}\right| \leq \frac{|xy|}{(x^2)^{1/2}} = |y|$$

or that $|xy| \leq (x^2 + y^2)/2$.

13. 0, see Exercise 11.

15. Let $\mathbf{x}$ take the role of $\mathbf{x}_0$ and $\mathbf{x} + \mathbf{h}$ that of $\mathbf{x}$ in the definition.

17. The vector $\mathbf{a}$ takes the place of $\mathbf{x}_0$ in the definition of a limit or in Theorem 6. In either case, the limit depends only on values of $f(\mathbf{x})$ for $\mathbf{x}$ near $\mathbf{x}_0$, not for $\mathbf{x} = \mathbf{x}_0$. Therefor $f(\mathbf{x}) = g(\mathbf{x})$ for $\mathbf{x} \neq \mathbf{a}$ certainly suffices to make the limits equal.

19. (a) $\lim_{\mathbf{x}\to\mathbf{0}}(f_1+f_2)(\mathbf{x})/\|\mathbf{x}\| = \lim_{\mathbf{x}\to\mathbf{0}} f_1(\mathbf{x})/\|\mathbf{x}\|+\lim_{\mathbf{x}\to\mathbf{0}} f_1(\mathbf{x})/\|\mathbf{x}\| = 0$.

    (b) Let $\epsilon > 0$. Since f is $o(\mathbf{x})$, there is a $\delta > 0$ such that $\|f(\mathbf{x})/\|\mathbf{x}\|\| < \epsilon/c$ whenever $0 < \|\mathbf{x}\| < \delta$. Then $\|(gf)(\mathbf{x})/\|\mathbf{x}\|\| \leq \|g(\mathbf{x})\|f(\mathbf{x})/\|\mathbf{x}\|\| < \epsilon$, so $\lim_{\mathbf{x}\to\mathbf{0}}(gf)(\mathbf{x})/\|\mathbf{x}\| = 0$.

    (c) $\lim_{x\to 0} f(x)/|x| = \lim_{x\to 0} |x| = 0$, so that $f(x)$ is $o(x)$. But $\lim_{x\to 0} g(x)/|x|$ does not exist, since $g(x)/|x| = \pm 1$ (as x is positive or negative). Therefore $g(x)$ is not $o(x)$.

Solutions to Selected Exercises in §2.7

1. Here, we have $f(x, y, z) = (e^x, \cos y, \sin z) = (f_1, f_2, f_3)$, so

$$\mathbf{D}f(x, y, z) = \begin{bmatrix} \partial f_1/\partial x & \partial f_1/\partial y & \partial f_1/\partial z \\ \partial f_2/\partial x & \partial f_2/\partial y & \partial f_2/\partial z \\ \partial f_3/\partial x & \partial f_3/\partial y & \partial f_3/\partial z \end{bmatrix} = \begin{bmatrix} e^x & 0 & 0 \\ 0 & -\sin y & 0 \\ 0 & 0 & \cos z \end{bmatrix}$$

$\mathbf{D}f$ is a diagonal matrix when f_1 depends only on the first variable, f_2 depends only on the second, and so on. Thus, $\mathbf{D}f$ is diagonal if f_n is a function of the n^{th} variable only.

4. A uniformly continuous function is not only continuous at all points in the domain, but more importantly, for given $\varepsilon > 0$, we can find *one* δ for all $\mathbf{x}_0$. This is different from continuity in that for continuity, we may find a δ which will work for a particular $\mathbf{x}_0$. A continuous function does not always have to be uniformly continuous (for examples: $f(x) = 1/x^2$ on $\mathbb{R}$ or $g(x) = 1/x$ on $(0, 1]$.)

(a) $\mathbf{T} : \mathbb{R}^n \to \mathbb{R}^m$ is linear implies that $\|\mathbf{T}\mathbf{x}\| \leq M\|\mathbf{x}\|$. Given $\varepsilon > 0$, $\mathbf{x}$ and $\mathbf{y} \in \mathbb{R}^n$, we have $\|\mathbf{T}(\mathbf{x} - \mathbf{y})\| \leq M\|\mathbf{x} - \mathbf{y}\|$ from exercise 2(a). Since $\mathbf{T}$ is linear, $\|\mathbf{T}(\mathbf{x}-\mathbf{y})\| \leq \|\mathbf{T}(\mathbf{x})-\mathbf{T}(\mathbf{y})\|$. Let $\delta = \varepsilon/M$; then for $0 \leq \|\mathbf{x} - \mathbf{y}\| \leq \delta$, we have $\|\mathbf{T}(\mathbf{x}) - \mathbf{T}(\mathbf{y})\| \leq M\delta = \varepsilon$. Note that this proof works because δ and ε do not depend on a particular choice of a point in $\mathbb{R}^n$.

(b) Given $\varepsilon > 0$, we want a $\delta > 0$ such that $0 < |x - x_0| < \delta$ implies $|1/x^2 - 1/{x_0}^2| < \varepsilon$ for x and x_0 in $(0, 1]$. We calculate:

$$\left|\frac{1}{x^2} - \frac{1}{{x_0}^2}\right| = \frac{|{x_0}^2 - x^2|}{x^2{x_0}^2} = \frac{|x - x_0||x + x_0|}{x^2{x_0}^2}$$
$$= |x - x_0| \cdot \left|\frac{1}{x{x_0}^2} + \frac{1}{x_0x^2}\right| < |x - x_0|\,\frac{2}{{x_0}^3} < \varepsilon$$

(if x_0 is the smaller of x and x_0), i.e.,

$$\left|\frac{1}{x^2} - \frac{1}{{x_0}^2}\right| \leq |x - x_0|\,\frac{2}{\{\min(x, x_0)\}^3}.$$

Note that $\min(x, x_0) > 0$. Let $\delta < (\varepsilon/2)\ \{\min(x, x_0)\}^3$, then $|x - x_0| < \delta$ implies

$$\left|\frac{1}{x^2} - \frac{1}{{x_0}^2}\right| < \delta \cdot \frac{2}{{x_0}^3} < \frac{\varepsilon}{2}{x_0}^3 \cdot \frac{2}{{x_0}^3} = \epsilon.$$

We have only shown that $f(x) = 1/x^2$ is continuous. It is not uniformly continuous since

$$\left|\frac{1}{x^2} - \frac{1}{{x_0}^2}\right| = \frac{\left|{x_0}^2 - x^2\right|}{x^2{x_0}^2} = \frac{|x - x_0||x + x_0|}{x^2{x_0}^2},$$

so for fixed ε, any δ approaches ∞ as x_0 goes to 0.

7. We have $f(x, y) = g(x) + h(y)$. What we want to prove is that as matrices,

$$\mathbf{D}f(x, y) = (\mathbf{D}g(x), \mathbf{D}h(y)).$$

Since g and h are differentiable at x_0 and y_0, respectively, we have

$$\lim_{x \to x_0} \frac{|g(x) - g(x_0) - \mathbf{D}g(x - x_0)|}{|x - x_0|} = 0$$

and

$$\lim_{y \to y_0} \frac{|h(y) - h(y_0) - \mathbf{D}h(y - y_0)|}{|y - y_0|} = 0$$

Now, use the triangle inequality:

$$\begin{aligned}&|g(x) - g(x_0) + h(y) - h(y_0) - (\mathbf{D}g(x - x_0) + \mathbf{D}h(y - y_0))| \\ &\quad \le |g(x) - g(x_0) - \mathbf{D}g(x - x_0)| + |h(y) - h(y_0) - \mathbf{D}h(y - y_0)|.\end{aligned}$$

Since $\|(x, y) - (x_0, y_0)\| \ge \|x - x_0\|$ and $\|(x, y) - (x_0, y_0)\| \ge \|y - y_0\|$, the sum of the two limits is greater than

$$\lim_{(x,y) \to (x_0,y_0)} \frac{|g(x) + h(y) - (g(x_0) + h(y_0)) - (\mathbf{D}g(x - x_0) + \mathbf{D}h(y - y_0))|}{\|(x, y) - (x_0, y_0)\|}.$$

Hence this limit goes to 0, and this satisfies the definition for differentiability of f at (x_0, y_0).

11. Given ε, note that

$$\left|\frac{xy}{(x^2 + y^2)^{1/2}} - 0\right| \le \left|\frac{xy}{x}\right| = |y|,$$

and we also know that $|y| \le \sqrt{x^2 + y^2}$. Let $\delta = \varepsilon$; then $\|(x, y) - (0, 0)\| = \sqrt{x^2 + y^2} < \delta$ implies that

$$\left|\frac{xy}{(x^2 + y^2)^{1/2}} - 0\right| < |y| < \varepsilon.$$

Therefore, $\lim_{(x,y) \to (0,0)} f(x, y) = 0$ and f is continuous.

16. (a) Suppose $\mathbf{y}$ is a boundary point of the open set $A \subset \mathbb{R}^n$. Then every ball centered at $\mathbf{y}$ contains points in A and points not in A. Therefore, the intersection of A and the ball $D_\varepsilon(\mathbf{y})$ is not empty. Hence

    (i) $\mathbf{y}$ is not in A since points of A have neighborhoods contained in A, and

    (ii) $\mathbf{y}$ is the limit of the sequence from A; choose $\varepsilon = 1/n$, $n = 1, 2, 3, \ldots$, then $D_{(1/n)}(\mathbf{y}) \cap A \neq \varnothing$ for $n = 1, 2, 3, \ldots$.

    Pick $\mathbf{x}_n$ to be an element of $D_{(1/n)}(\mathbf{y}) \cap A$. Then $\mathbf{x}_n$ is in A and $\|\mathbf{y} - \mathbf{x}_n\| \leq 1/n$, which goes to zero as n goes to infinity. Suppose $\mathbf{y}$ is not in A, but there is a sequence $\{\mathbf{x}_n\}$ in A converging to $\mathbf{y}$. Let $\varepsilon > 0$. The $\mathbf{y}$ is in the intersection of $D_\varepsilon(\mathbf{y})$ and the set of all points not in A, so the right-hand side is not empty. On the other hand, there exists an N such that $n \geq N$ implies that $\|\mathbf{x}_n - \mathbf{y}\| < \epsilon$, i.e., $n \geq N$ implies that $\mathbf{x}_n$ is in $D_\varepsilon(\mathbf{y})$. $\mathbf{x}_N$ is in A, so $\mathbf{x}_N$ is in $D_\varepsilon(\mathbf{y}) \cap A$. Hence, the right-hand side is a boundary point of A.

    (b) Suppose $\lim_{\mathbf{x} \to \mathbf{y}} f(\mathbf{x}) = b$, i.e., for every $\varepsilon > 0$, there is a $\delta > 0$ such that for $\mathbf{x}$ in A, $\|\mathbf{x} - \mathbf{y}\| < \delta$ implies that $\|f(\mathbf{x}) - f(\mathbf{y})\| < \epsilon$. Let $\{\mathbf{x}_n\}$ be a sequence in A converging to $\mathbf{y}$. Fix $\varepsilon > 0$. Choose $\delta > 0$ as above. Choose N so that $n \geq N$ implies that $\|\mathbf{x}_n - \mathbf{y}\| < \delta$. Then $n \geq N$ implies $\|f(\mathbf{x}_n) - b\| < \epsilon$, so the sequence $|f(\mathbf{x}_n)|$ converges to b. (We need $\mathbf{y}$ to be on the boundary of A to guarantee the existence of the sequence $\{\mathbf{x}_n\}$.)

    To go the other way, suppose it is not the case that $\lim_{\mathbf{x} \to \mathbf{y}} f(\mathbf{x}) = b$. Then there exist $\varepsilon > 0$ such that for every $\delta > 0$, there exists an $\mathbf{x}$ in A with $\|\mathbf{x} - \mathbf{y}\| < \delta$, but $\|f(\mathbf{x}) - b\| > \varepsilon$. Choose $\mathbf{x}_n$ in A such that $\|\mathbf{x}_n - \mathbf{y}\| < 1/n$, but $\|f(\mathbf{x}_n) - b\| > \epsilon$ for $n = 1, 2, 3, \ldots$ (corresponding to $\delta = 1/n$). Then $\mathbf{x}_n$ converges to $\mathbf{y}$ (as in part (a)), but $f(\mathbf{x})$ does not converge to b. (Note: We have proved the "contrapositive" of the theorem. We *negate* the statement of the hypothesis (note the changes in "there exists," "for every," and the inequality signs), and prove the *negation* of the result we want to arrive at. The difference between this method and the method of proving by contradiction is that we do *not* negate the hypothesis.)

    (c) Let U be open in $\mathbb{R}^m$ with $\mathbf{x}$ in U. The proof follows from part (b). Specifically,

$$\lim_{x_n \to x} f(\mathbf{x}_n) = f(\mathbf{x}) \text{ implies } f(\mathbf{x}_n) \to f(\mathbf{x})$$

    for any sequence $\mathbf{x}_n$ converging to $\mathbf{x}$ in U. By part (b), f is continuous at $\mathbf{x}$.

§3.4A: Second Derivative Test: Constrained Extrema

1. It is the usual second derivative test in one-variable calculus.

§4.1A: Equilibria in Mechanics

1. $(-\frac{1}{4}, -\frac{1}{4})$
3. $(2, 1)$ is an unstable equilibrium.
5. Stable equilibrium point $(2 + m^2g^2)^{-1/2}(-1, -1, -mg)$

Solution to Exercise 3. By Theorem 14, the critical points of the potential V are the equilibrium points. These points are where the gradient vanishes. In fact,

$$\nabla V(x, y) = (2x + 4y - 8)\mathbf{i} + (4x - 2y - 6)\mathbf{j} = \mathbf{0}$$

only if $2x + 4y - 8 = 0$ and $4x - 2y - 6 = 0$. Simplifying we get the system of equations

$$\begin{aligned} x + 2y &= 4 \\ 2x - y &= 3 \end{aligned}$$

whose solution is $x = 2, y = 1$. This critical point $(2, 1)$ will be stable if it is a strict local minimum for V. Using the second test (Theorem 6), we have

$$\frac{\partial^2 V}{\partial x^2}(2, 1) = 2, \quad \frac{\partial^2 V}{\partial y^2}(2, 1) = -2, \quad \frac{\partial^2 V}{\partial x \partial y}(2, 1) = 4.$$

Therefore the discriminant

$$D = \left(\frac{\partial^2 V}{\partial x^2}\right)\left(\frac{\partial^2 V}{\partial y^2}\right) - \left(\frac{\partial^2 V}{\partial x \partial y}\right)^2 = -20 < 0.$$

The second derivative test thus tells us that the point is a saddle; in particular, we cannot conclude that the point is stable. (In fact, it is unstable, but this has not been discussed carefully, so it is best to just say the stability test fails). ♦

§4.1B Rotations and the Sunshine Formula

1. (a) $\mathbf{m}_0 = (1/\sqrt{6})(\mathbf{i} + \mathbf{j} + 2\mathbf{k})$,
   $\mathbf{n}_0 = (1/2\sqrt{3})(\mathbf{i} + \mathbf{j} - \mathbf{k})$
   (b) $\mathbf{r} = [\cos(\pi t/12)/2\sqrt{2} + \sin(\pi t/12)/4]\mathbf{i} + [-1/2\sqrt{2} + \cos(\pi t/12)/2\sqrt{2} + \sin(\pi t/12)/4]\mathbf{j} + [-1/2\sqrt{2} + \cos(\pi t/12)/\sqrt{2} - \sin(\pi t/12)/4]\mathbf{k}$

(c) $(x, y, z) = (-1/2\sqrt{2})(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) + (-\pi/48)(\mathbf{i} + \mathbf{j} + \mathbf{k})(t - 12)$

3. T_d would be longer.

5. The "exact" formula is $-\tan l \sin \alpha = \cos(2\pi t/T_d)[\tan(2\pi t/T_y) \tan(2\pi t/T_d) - \cos \alpha]$.

7. $A = 9.4°$

9. The equator would receive approximately six times as much solar energy as Paris.

§4.4 Flows and the Geometry of the Divergence

1. If $\mathbf{x} = (x_1, x_2, x_3)$, $\phi(\mathbf{x}, t) = (\phi_1, \phi_2, \phi_3)$, and $f = f(x_1, x_2, x_3, t)$, then by the chain rule,

$$\begin{aligned} \frac{d}{dt}(f(\phi(\mathbf{x}, t), t)) &= \frac{\partial f}{\partial t}(\mathbf{x}, t) + \sum_{i=1}^{3} \frac{\partial f}{\partial x_i}(\phi(\mathbf{x}, t), t) \frac{\partial \phi_i}{\partial t}(\mathbf{x}, t) \\ &= \frac{\partial f}{\partial t}(\mathbf{x}, t) + [\nabla f(\phi(\mathbf{x}, t), t)] \cdot [\mathbf{F}(\phi(\mathbf{x}, t))]. \end{aligned}$$

3. By the product rule,

$$\frac{d}{dt}\mathbf{v} \cdot \mathbf{w} = \frac{d\mathbf{v}}{dt} \cdot \mathbf{w} + \mathbf{v} \cdot \frac{d\mathbf{w}}{dt}.$$

Substitute equation (2) into this. For the last equality, use the identity $A^T\mathbf{v} \cdot \mathbf{w} = \mathbf{v} \cdot A\mathbf{w}$.

5. By the choice of axes,

$$\mathbf{w} = \frac{1}{2}(\nabla \times \mathbf{F})(\mathbf{0}) = \omega\mathbf{k}.$$

From the text,

$$\mathbf{v} = -\omega y\mathbf{i} + \omega x\mathbf{j}$$

and therefore

$$\mathbf{Dv}(\mathbf{0}) = \begin{bmatrix} \mathbf{b}0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

On the other hand, by the definitions of W and $\nabla \times \mathbf{F}$,

$$
\begin{aligned}
W &= \begin{bmatrix} \mathbf{b}w_1 1 & w_1 2 & w_1 3 \\ w_2 1 & w_2 2 & w_2 3 \\ w_3 1 & w_3 2 & w_3 3 \end{bmatrix} \\
&= \begin{bmatrix} 0 & \frac{1}{2}\left(\frac{\partial F_1}{\partial y} - \frac{\partial F_2}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) & 0 & \frac{1}{2}\left(\frac{\partial F_2}{\partial z} - \frac{\partial F_3}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial y}\right) & \frac{1}{2}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) & 0 \end{bmatrix} \\
&= \frac{1}{2}\begin{bmatrix} 0 & -(\nabla \times \mathbf{F})_z & (\nabla \times \mathbf{F})_y \\ (\nabla \times \mathbf{F})_z & 0 & -(\nabla \times \mathbf{F})_x \\ -(\nabla \times \mathbf{F})_y & (\nabla \times \mathbf{F})_x & 0 \end{bmatrix}
\end{aligned}
$$

Our choice of coordinate axes gives

$$
W = \begin{bmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
$$

To interpret the result, we note that the vector field $\mathbf{v}$ represents rotation around a fixed axis $\mathbf{w}$. The flow $\psi(\mathbf{x}, t)$ of $\mathbf{v}$ rotates points in this field, and, for fixed t, its derivative $\mathbf{D}_{\mathbf{x}}\psi(\mathbf{x}, t)$ rotates vectors as well. Let $\mathbf{Y}$ be an arbitrary vector and set $\mathbf{Y}(t) = \mathbf{D}_{\mathbf{x}}\psi(\mathbf{x}, t)\mathbf{Y}$. As t increases or decreases, $\mathbf{Y}(t)$ rotates around $\mathbf{w}$ and

$$
\left.\frac{d\mathbf{Y}}{dt}\right|_{t=0} = \mathbf{D}_{\mathbf{x}}\mathbf{v}(\mathbf{0})\mathbf{Y}.
$$

This gives the rate of change of $\mathbf{Y}$ as it is transported (rotated) by $\mathbf{D}_{\mathbf{x}}\psi$. By Exercise 3, the rate of change of any vector $\mathbf{x}$ at the origin under transport by the derivative of the flow $\phi(\mathbf{x}, t)$ of $\mathbf{F}$ is given by

$$
\left.\frac{d\mathbf{x}}{dt}\right|_{t=0} = \mathbf{D}_{\mathbf{x}}\mathbf{F}(0)\mathbf{x} = (S + W)\mathbf{x}.
$$

Thus this rate of change of $\mathbf{x}$ has two components: the deformation matrix, which affects inner products, and the W matrix. Thus the *W matrix is precisely the rate of change of vectors as they undergo an infinitesimal rotation around the axis* $(\operatorname{curl}\mathbf{F})(\mathbf{0}) = (\nabla \times \mathbf{F})(\mathbf{0})$ by the mapping $\mathbf{D}_{\mathbf{x}}\psi(\mathbf{x}, t)$.

[The deformation matrix S incorporates all the length and angle changes caused by the flow. In particular, volume changes are contained in S. In fact, the trace of S is the divergence; $\operatorname{tr} S = \operatorname{div}\mathbf{F}(\mathbf{x})$.

(The trace of a matrix is the sum of its diagonal entries). The trace-free part of S,

$$S' = S - \frac{1}{3}(trS^I)$$

where I is the 3×3 identity, is called the *shear*.]

7. The line $\mathbf{x} + \lambda\mathbf{v}$ is carried to the curve $\lambda \mapsto \phi(\mathbf{x} + \lambda\mathbf{v}, t)$ after time t, which for λ small, is approximated by its tangent line, namely, $\lambda \mapsto \phi(\mathbf{x}, t) + \mathbf{D_x}\phi(\mathbf{x}, t) \cdot \lambda\mathbf{v}$.

§5.6 Some Technical Integration Theorems

1. If $a \neq b$, let $\epsilon = |a - b|/2$.

3. Let $e = 2d - c$, so that $d = (c+e)/2$. Consider the vertical "doubling" of R defined by $Q = R \cup R_1$, where $R_1 = [a, b] \times [d, e]$. If f is extended to Q by letting f be 0 on the added part, then f is integrable over Q by additivity. The nth regular partition of $[(a+b)/2, b] \times [c, d]$ is part of the $2n$th regular partition of Q. For large n, the Riemann sums for that $2n$th partition cannot vary by more than ϵ as we change the points from the subrectangles, in particular if we change only those in $[(a+b)/2, b] \times [c, d]$. These changes correspond to the possible changes for the Riemann sums for the nth partition of $[(a + b)/2, b] \times [c, d]$. The argument for part (b) is similar.

5. Let $R = [a, b] \times [c, d]$ and $B = [e, f] \times [g, h]$. Since the rectangles of a partition of R intersect only along their edges, their areas can be added, and b_n is the area of the union of all subrectangles of the nth regular partition of R that intersect B. Since B is contained in this union, area $(B) \leq b_n$. On the other hand, if (x, y) is in the union, then

$$e - (b - a)/n \leq x \leq f + (b - a)/n$$

and

$$g - (d - c)/n \leq y \leq h + (d - c)/n.$$

This leads to

$$b_n \leq \text{area}(B) + 2[(b-a)(h-g) + (d-c)(f-e)]/n + 4(a-b)(d-c)/n^2.$$

Letting $n \to \infty$ and combining the inequalities proves the assertion.

7. (a) The strategy is to go from point to point within $[a, b]$ by short steps, adding up the changes as you go. Given $\epsilon > 0, \phi$ is uniformly continuous and therefore there exists a $\delta > 0$ such that $|\phi(x) - \phi(y)| \leq \epsilon$ whenever $|x - y| < \delta$. Let $x \in [a, b]$ and introduce intermediate points $a = x_0 < x_1 < \ldots < x_{n-1} < x_n =$

x with $x_{i+1} - x_i < \delta$. This can be done with no more than $[(b-a)/\delta] + 1$ segments. By the triangle inequality,

$$|\phi(x) - \phi(a)| \leq \sum_{i=1}^{n} |\phi(x_i) - \phi(x_{i-1})| \leq \left(\frac{b-a}{\delta+1}\right)\epsilon.$$

Thus $|\phi(x)| \leq |\phi(a)| + [(b-a)/(\delta+1)]\epsilon$ for every x in $[a, b]$.

(b) Use an argument like that for part (a), moving by short steps within the rectangle $[a, b] \times [c, d]$.

(c) This is trickier, since D may be composed of many disconnected pieces so that the short steps cannot be taken within D. Nevertheless, given ϵ, there is a δ such that$|f(\mathbf{x}) - f(\mathbf{y})| \leq \epsilon$ whenever $\mathbf{x}$ and $\mathbf{y}$ are in D and $\|\mathbf{x} - \mathbf{y}\| < \delta$, by the uniform boundedness principle. Since D is bounded, we may find a large "cube" R with sides of length L such that $D \subset R$. Partition R into subcubes by dividing each edge into m parts. The diagonal of each subcube has length $\sqrt{n}L/m$. If we take $m > \sqrt{n}L/\delta$, any two points in the same subcube are less then δ apart, and there are m^n subcubes. If $R_1, \ldots, R_n$ are those that intersect D, choose $\mathbf{x}_i \in D \cap R_i$. For any $\mathbf{x} \in D$, we have $|f(\mathbf{x}) < \epsilon + \text{map}(|f(\mathbf{x}_1)|, \ldots, |f(\mathbf{x})|)$.

Solution to Exercise 2(a).

Exercise 2(a). *Let f be the function on the half open interval $(0, 1]$ defined by $f(x) = 1/x$. Show that f is continuous at every point of $(0, 1]$ but not uniformly continuous.*

Solution. Let $x_0 \in (0, 1]$. Given $\epsilon > 0$, we must find a $\delta > 0$ such that whenever

$$0 < |x - x_0| < \delta, \quad x\epsilon(0, 1], \quad \text{then} \quad \left|\frac{1}{x} - \frac{1}{x_0}\right| < \epsilon.$$

Notice that

$$\left|\frac{1}{x} - \frac{1}{x_0}\right| = \frac{|x - x_0|}{|xx_0|}.$$

If $|x - x_0| < x_0/2$, then $x > x_0/2 > 0$ and

$$\frac{|x - x_0|}{|xx_0|} < \frac{2|x - x_0|}{x_0^2}.$$

If $|x - x_0|$ is also less than $\epsilon \cdot x_0^2/2$, then $2|x - x_0|/x_0^2 < \epsilon$. Thus by picking $\delta < \min\left(x_0/2, (\epsilon/2) \cdot x_0^2\right)$ one has

$$\left|\frac{1}{x} - \frac{1}{x_0}\right| < \epsilon.$$

Thus, f is continuous at x_0. To show that f is not uniformly continuous, we suppose it were and derive a contradiction. Let $\epsilon = 1/2$. If f were uniformly continuous there exists a $\delta > 0$ such that

$$\left|\frac{1}{x_1} - \frac{1}{x_2}\right| < \frac{1}{2}$$

whenever $|x_2 - x_1| < \delta$. Let $N > 2\delta$. There for all integers $n, m > N$

$$\left|\frac{1}{n} - \frac{1}{m}\right| < \frac{1}{N} + \frac{1}{N} = \frac{2}{N} < \delta.$$

Set $x_1 = 1/n$ and $x_2 = 1/(n+1)$. Then

$$\left|\frac{1}{x_1} - \frac{1}{x_2}\right| = \frac{1}{\frac{1}{n+1}} - \frac{1}{\frac{1}{n}} = 1.$$

But 1 is *not* less than 1/2 a contradiction. ♦

§8.5 Green's Functions

1. Write the components of φ as $\xi(\mathbf{x}, t), \eta(\mathbf{x}, t)$, and $\zeta(\mathbf{x}, t)$. First, observe that by definition of φ,

$$\frac{\partial}{\partial t}\varphi(\mathbf{x}, t) = \mathbf{F}(\varphi(\mathbf{x}, t), t).$$

The determinant J can be differentiated by recalling that the determinant of a matrix is multilinear in the columns (or rows). Thus, holding $\mathbf{x}$ fixed,

$$\frac{\partial}{\partial t}J = \begin{bmatrix} \frac{\partial}{\partial t}\frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x} & \frac{\partial \zeta}{\partial x} \\ \frac{\partial}{\partial t}\frac{\partial \xi}{\partial y} & \frac{\partial \eta}{\partial y} & \frac{\partial \zeta}{\partial y} \\ \frac{\partial}{\partial t}\frac{\partial \xi}{\partial z} & \frac{\partial \eta}{\partial z} & \frac{\partial \zeta}{\partial z} \end{bmatrix} + \begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial}{\partial t}\frac{\partial \eta}{\partial x} & \frac{\partial \zeta}{\partial x} \\ \frac{\partial \xi}{\partial y} & \frac{\partial}{\partial t}\frac{\partial \eta}{\partial y} & \frac{\partial \zeta}{\partial y} \\ \frac{\partial \xi}{\partial z} & \frac{\partial}{\partial t}\frac{\partial \eta}{\partial z} & \frac{\partial \zeta}{\partial z} \end{bmatrix} + \begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x} & \frac{\partial}{\partial t}\frac{\partial \zeta}{\partial x} \\ \frac{\partial \xi}{\partial y} & \frac{\partial \eta}{\partial y} & \frac{\partial}{\partial t}\frac{\partial \zeta}{\partial y} \\ \frac{\partial \xi}{\partial z} & \frac{\partial \eta}{\partial z} & \frac{\partial}{\partial t}\frac{\partial \zeta}{\partial z} \end{bmatrix}.$$

Now write

$$\begin{aligned} \frac{\partial}{\partial t}\frac{\partial \xi}{\partial x} &= \frac{\partial}{\partial x}\frac{\partial \xi}{\partial t} = \frac{\partial}{\partial x}F_1(\varphi(\mathbf{x}, t), t), \\ \frac{\partial}{\partial t}\frac{\partial \xi}{\partial y} &= \frac{\partial}{\partial y}\frac{\partial \xi}{\partial t} = \frac{\partial}{\partial y}F_2(\varphi(\mathbf{x}, t), t), \\ \frac{\partial}{\partial t}\frac{\partial \zeta}{\partial z} &= \frac{\partial}{\partial z}\frac{\partial \zeta}{\partial t} = \frac{\partial}{\partial z}F_3(\varphi(\mathbf{x}, t), t). \end{aligned}$$

The components F_1, F_2, and F_3 of $\mathbf{F}$ in this expression are functions of x, y, and z through $\varphi(\mathbf{x}, t)$; therefore,

$$\frac{\partial}{\partial x} F_1(\varphi(\mathbf{x},t),t) = \frac{\partial F_1}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial F_1}{\partial \eta}\frac{\partial \eta}{\partial x} + \frac{\partial F_1}{\partial \zeta}\frac{\partial \zeta}{\partial x},$$

$$\vdots$$

$$\frac{\partial}{\partial z} F_3(\varphi(\mathbf{x},t),t) = \frac{\partial F_3}{\partial \xi}\frac{\partial \xi}{\partial z} + \frac{\partial F_3}{\partial \eta}\frac{\partial \eta}{\partial z} + \frac{\partial F_3}{\partial \zeta}\frac{\partial \zeta}{\partial z}.$$

When these are substituted into the previous expression for $\partial J/\partial t$, one gets for the respective terms

$$\frac{\partial F_1}{\partial x} J + \frac{\partial F_2}{\partial y} J + \frac{\partial F_3}{\partial z} J = (\operatorname{div} \mathbf{F}) J.$$

3. HINTS: By the transport equation from Theorem 12, with $\mathbf{V}$ in place of $\mathbf{F}$,

$$\frac{d}{dt} \iiint_{W_t} \rho \, dx\, dy\, dz = \iiint_{W_t} \left(\frac{D\rho}{Dt} + \rho \operatorname{div} \mathbf{V} \right) dx\, dy\, dz.$$

Now use the fact that

$$\frac{D\rho}{Dt} + \rho \operatorname{div} \mathbf{V} = \operatorname{div} \mathbf{J} + \frac{\partial \rho}{\partial t},$$

where $\mathbf{J} = \rho \mathbf{V}$, as in the text.

5. If v_i is the ith component of a vector $\mathbf{v}$, then by the transport equation (Exercise 2),

$$\begin{aligned}
\left[\frac{d}{dt} \iiint_{W_t} f\mathbf{F}\, dx\, dy\, dz \right]_i &= \frac{d}{dt} \iiint_{W_t} (f\mathbf{F})_i \, dx\, dy\, dz = \frac{d}{dt} \iiint_{W_t} fF_i \, dx\, dy\, dz \\
&= \iiint_{W_t} \left[\frac{D(fF_i)}{Dt} + (fF_i) \operatorname{div} \mathbf{F} \right] dx\, dy\, dz \\
&= \iiint_{W_t} \left[\frac{\partial}{\partial t}(fF_i) + \mathbf{D}_x(fF_i) \cdot \mathbf{F} + (f\mathbf{F}_i) \operatorname{div} \mathbf{F} \right] dx\, dy\, dz \\
&= \iiint_{W_t} \left[\frac{\partial}{\partial t}(f\mathbf{F}_i) + \nabla(fF_i) \cdot \mathbf{F} + (fF_i) \operatorname{div} \mathbf{F} \right] dx\, dy\, dz \\
&= \iiint_{W_t} \left\{ \frac{\partial}{\partial t}(f\mathbf{F}_i) + [\mathbf{D}(f\mathbf{F})\mathbf{F}]_i + [(f\mathbf{F}) \operatorname{div} \mathbf{F}]_i \right\} dx\, dy\, dz \\
&= \iiint_{W_t} \left[\frac{\partial}{\partial t}(f\mathbf{F}) + \mathbf{D}(f\mathbf{F})\mathbf{F} + (f\mathbf{F}) \operatorname{div} \mathbf{F} \right]_i dx\, dy\, dz \\
&= \left[\iiint_{W_t} \frac{\partial}{\partial t}(f\mathbf{F}) + \mathbf{D}(f\mathbf{F})\mathbf{F} + (f\mathbf{F}) \operatorname{div} \mathbf{F}\, dx\, dy\, dz \right]_i \\
&= \left[\iiint_{W_t} \left(\frac{\partial}{\partial t}(f\mathbf{F}) + (\mathbf{F} \cdot \nabla)(f\mathbf{F}) + (f\mathbf{F}) \operatorname{div} \mathbf{F} \right) dx\, dy\, dz \right]_i .
\end{aligned}$$

7. (a) Because $\mathbf{V} = \nabla\phi, \nabla\times\mathbf{V} = \mathbf{0}$, and therefore $(\mathbf{V}\cdot\nabla)\mathbf{V} = \dfrac{1}{2}\nabla(\|\mathbf{V}\|^2)$, Euler's equation becomes

$$-\frac{\nabla p}{\rho} = \frac{d\mathbf{V}}{dt} + \frac{1}{2}\nabla(\|\mathbf{V}\|^2) = \nabla\Big(\frac{d\phi}{dt} + \frac{1}{2}\|\mathbf{V}\|^2\Big).$$

If $\mathbf{c}$ is a path from P_1 to P_2, then

$$\begin{aligned}-\int_{\mathbf{c}}\frac{1}{\rho}\,dp = -\int\frac{1}{\rho}\nabla p\cdot\mathbf{c}'(t)\,dt &= \int_{\mathbf{c}}\nabla\Big(\frac{d\phi}{dt} + \frac{1}{2}\|\mathbf{V}\|^2\Big)\cdot\mathbf{c}'(t)\,dt\\ &= \Big(\frac{d\phi}{dt} + \frac{1}{2}\|\mathbf{V}\|^2\Big)\Big|_{P_1}^{P_2}.\end{aligned}$$

(b) If $d\mathbf{V}/dt = \mathbf{0}$ and ρ is constant, then $\frac{1}{2}\nabla(\|\mathbf{V}\|^2) = -(\nabla p)/\rho = -\nabla(p/\rho)$, and therefore $\nabla\left(\dfrac{1}{2}\|\mathbf{V}\|^2 + p/\rho\right) = \mathbf{0}$.

9. By Ampère's law, $\nabla\cdot\mathbf{J} = \nabla\cdot(\nabla\times\mathbf{H}) - \nabla\cdot(\partial\mathbf{E}/\partial t) = -\nabla\cdot(\partial\mathbf{E}/\partial t) = -(\partial/\partial t)(\nabla\cdot\mathbf{E})$. By Gauss' law this is $-\partial\rho/\partial t$. Thus, $\nabla\cdot\mathbf{J} + \partial\rho/\partial t = 0$.

10. (a) If $\mathbf{x}\in S$, then $r''a/R = r$, so $G = 0$. In general, $r = \|\mathbf{x}-\mathbf{y}\|$ and $r'' = \|\mathbf{x}-\mathbf{y}\|$, and therefore

$$G = \frac{1}{4\pi}\left(\frac{R}{a}\frac{1}{\|\mathbf{x}-\mathbf{y}''\|} - \frac{1}{\|\mathbf{x}-\mathbf{y}\|}\right) \quad \text{and}$$

$$\nabla_{\mathbf{x}}G = \frac{1}{4\pi}\left(\frac{R}{a}\frac{\mathbf{y}'-\mathbf{x}}{\|\mathbf{x}-\mathbf{y}'\|^3} - \frac{\mathbf{y}-\mathbf{x}}{\|\mathbf{x}-\mathbf{y}\|^3}\right)$$

and $\nabla_{\mathbf{x}}^2 G = 0$ when $\mathbf{x}\neq\mathbf{y}$, just as in the analysis of equation (15)($\mathbf{x}\neq\mathbf{y}'$, since $\mathbf{x}$ is inside and $\mathbf{y}'$ is outside the sphere). Theorem 10 gives $\nabla^2 G = \delta(\mathbf{x}-\mathbf{y}) - (R/a)\delta(\mathbf{x}-\mathbf{y}')$, but the second term is always 0, since $\mathbf{x}$ is never $\mathbf{y}'$. Therefore $\nabla^2 G = \delta(\mathbf{x}-\mathbf{y})$ for $\mathbf{x}$ and $\mathbf{y}$ in the sphere.

(b) If $\mathbf{x}$ is on the surface of S, then $\mathbf{n} = \mathbf{x}/R$ is the outward unit normal, and

$$\begin{aligned}\frac{\partial G}{\partial\mathbf{n}} &= \frac{1}{4\pi}\left[\frac{R}{a}\nabla\left(\frac{1}{r''}\right)\cdot\mathbf{n} - \nabla\left(\frac{1}{r}\right)\cdot\mathbf{n}\right]\\ &= \frac{1}{4\pi}\left(\frac{R}{a}\frac{\mathbf{y}'-\mathbf{x}}{\|\mathbf{y}'-\mathbf{x}\|^3} - \frac{\mathbf{y}-\mathbf{x}}{\|\mathbf{x}-\mathbf{y}\|^3}\right)\cdot\mathbf{n}.\end{aligned}$$

If γ is the angle between $\mathbf{x}$ and $\mathbf{y}$, then $\|\mathbf{x}-\mathbf{y}\|^2 = r^2 = R^2 + a^2 - 2aR\cos\gamma$ and $\|\mathbf{x}-\mathbf{y}'\|^2 = r''^2 = R^2 + b^2 - 2bR\cos\gamma = (R^2/a^2)r^2$. Then

$$\frac{\partial G}{\partial\mathbf{n}} = \frac{1}{4\pi r^3}\left[\frac{R}{a}\frac{\mathbf{y}'-\mathbf{x}}{(R/a)^3} - (\mathbf{y}-\mathbf{x})\right]\cdot\mathbf{n}.$$

But $\mathbf{x}\cdot\mathbf{n} = R$ and $\mathbf{n} = \mathbf{x}/R$, and so this becomes

$$\begin{aligned}\frac{\partial G}{\partial \mathbf{n}} &= \frac{1}{4\pi^3}\left(\frac{a^2}{R^3}\mathbf{y}'\cdot\mathbf{x} - \frac{a^2}{R} - \frac{\mathbf{y}\cdot\mathbf{x}}{R} + R\right)\\ &= \frac{1}{4\pi^3 R}\left(\frac{a^2}{R^2}\|\mathbf{y}'\|R\cos\gamma - \|\mathbf{y}\|R\cos\gamma + R^2 - a^2\right)\\ &= \frac{R^2-a^2}{4\pi R}\frac{1}{r^3} \quad \text{since} \quad \|\mathbf{y}'\| = R^2/a.\end{aligned}$$

Integrating over the surface of the sphere,

$$\begin{aligned}u(\mathbf{y}) &= \iint_S f\frac{\partial G}{\partial \mathbf{n}}\,dS\\ &= \int_0^{2\pi}\int_0^{\pi}\left[f(\theta,\phi)\frac{R^2-a^2}{4\pi R}\frac{1}{r^3}R^2\sin\phi\right]\,d\phi d\theta\\ &= \frac{R(R^2-a^2)}{4\pi}\int_0^{2\pi}\int_0^{\pi}\frac{f(\theta,\phi)\sin\phi\;d\phi d\theta}{(R^2+a^2-2aR\cos\gamma)^{3/2}}.\end{aligned}$$

11. (a) According to equations (12) of this section, we need to show that

$$\tilde{G}(\mathbf{x},\mathbf{y}) = \tilde{G}(\mathbf{y},\mathbf{x}) \tag{1}$$

and

$$\nabla^2 G = \delta(\mathbf{x},\mathbf{y}). \tag{2}$$

Define $\mathbf{r} = \mathbf{x}-\mathbf{y}$ and $\mathbf{r}' = R(\mathbf{x}) - \mathbf{y}$. To show (1),

$$\tilde{G}(\mathbf{y},\mathbf{x}) = G(\mathbf{y},\mathbf{x}) - G(\mathbf{y},R(\mathbf{x})) = -\frac{1}{4}\pi\|\mathbf{y}-\mathbf{x}\| + \frac{1}{4}\pi\|\mathbf{y}-R(\mathbf{x})\|.$$

It is obvious that $\|\mathbf{r}\| = \|{-\mathbf{r}}\|$, so $\tilde{G}(\mathbf{y},\mathbf{x}) = \tilde{G}(\mathbf{x},\mathbf{y})$. To show (2), we know $\nabla G(\mathbf{x},\mathbf{y}) = \mathbf{r}/4\pi r^3$. Thus $\nabla G(R(\mathbf{x}),\mathbf{y}) = \mathbf{r}'/4\pi(r')^3$. Then $\nabla^2\tilde{G} = \delta(\mathbf{x},\mathbf{y}) - \delta(R(\mathbf{x}),\mathbf{y})$, and

$$\iiint_{\mathbb{R}^3}\nabla^2\tilde{G}\,d\mathbf{y} = \iiint_{\mathbb{R}^3}[\delta(\mathbf{x},\mathbf{y}) - \delta(R(\mathbf{x}),\mathbf{y})]\,d\mathbf{y}.$$

If $\mathbf{x} = \mathbf{y}$, then $R(\mathbf{x}) \neq \mathbf{y}$, and the above integral becomes

$$\iiint_{\mathbb{R}^3}\nabla^2\tilde{G}\,d\mathbf{y} = \iiint_{\mathbb{R}^3}\delta(\mathbf{x},\mathbf{y})\,d\mathbf{y} = 1;$$

If $R(\mathbf{x}) = \mathbf{y}$, then $\mathbf{x} \neq \mathbf{y}$. Make a change of variable: Let $u = x$, $v = y$, $w = -z$, so $d\mathbf{y} = dx\,dy\,dz = -du\,dv\,dw$. Now the integral becomes

$$\begin{aligned}\iiint_{\mathbb{R}^3}\nabla^2\tilde{G}\,d\mathbf{y} &= -\iiint_{\mathbb{R}^3}\delta(R(\mathbf{x}),\mathbf{y})\,d\mathbf{y}\\ &= -\iiint_{\mathbb{R}^3}\delta(R(\mathbf{x}),\mathbf{u})\,(-d\mathbf{u})\\ &= \iiint_{\mathbb{R}^3}\delta(R(\mathbf{x}),\mathbf{u})\,d\mathbf{u} = 1,\end{aligned}$$

where $\mathbf{u} = (u, v, w)$.

(b) Simply "stick" our Green's function into an integral:

$$\begin{aligned} u(\mathbf{x}) &= \iiint_{\mathbb{R}^3} \tilde{G}(\mathbf{x}, \mathbf{y})\rho(\mathbf{y})\, d\mathbf{y} \\ &= \iiint_{\mathbb{R}^3} G(\mathbf{x}, \mathbf{y})\rho(\mathbf{y}) - G(R(\mathbf{x}), \mathbf{y})\rho(\mathbf{y})\, d\mathbf{y}. \end{aligned}$$

12. (a) Use the Chain rule: $u_t + u_{xx} = u_t\dot{t} + u_x\dot{x} = \dot{u} = 0$.

(b) Use $dt/ds = 1$ and $dx/ds = u$. The slope is equal to

$$\frac{t(s)}{x(s)} = \frac{dt}{dx} = \frac{1}{u}.$$

From part (a), u is a constant, therefore $1/u$ is also a constant. Hence the characteristic curves are straight lines.

(c) Two characteristics through $(x_1, 0)$ and $(x_2, 0)$ have equations

$$t = [1/u_0(x_1)](x - x_1) \text{ and } t = [1/u_0(x_2)](x - x_2),$$

respectively. The intersection is

$$[1/u_0(x_1)](x - x_1) = [1/u_0(x_2)](x - x_2).$$

Simplify and get

$$x(1/u_0(x_1) - 1/u_0(x_2)) = x_1/u_0(x_1) - x_2/u_0(x_2).$$

Solve for x:

$$x = \frac{\frac{x_1}{u_0(x_1)} - \frac{x_2}{u_0(x_2)}}{\frac{1}{u_0(x_1)} - \frac{1}{u_0(x_2)}} = \frac{x_1u_0(x_2) - x_2u_0(x_1)}{u_0(x_2) - u_0(x_1)} > 0.$$

(d) Plug in:

$$\bar{t} = \frac{1}{u_0(x_1)}\left(\frac{x_1u_0(x_2) - x_2u_0(x_1)}{u_0(x_2) - u_0(x_1)} - x_1\right) = -\frac{x_2 + x_1}{u_0(x_2) - u_0(x_1)}$$

13. (a) $\dot{u} = (d/ds)[u(x(s), t(s))] = u_x\dot{x} + u_t\dot{t} = u_xf'(u) + u_t = 0$.

(b) If the characteristic curve $u(x, t) = c$ (by part (a)), define t implicitly as a function of x; then $u_x + u_t(dt/dx) = 0$. But also $u_t + f(u)_x = 0$; that is, $u_t + f'(u)u_x = 0$. These two equations together give $dt/dx = 1/f'(u) = 1/f'(c)$. Therefor the curve is a straight line with slope $1/f'(c)$.

(c) If $x_1 < x_2, u_0(x_1) > u_0(x_2) > 0$, and $f'(u_0(x_2))$, then $f'(u_0(x_1)) > f'(u_0(x_2)) > 0$, since $f'' > 0$. The characteristic through $(x_1, 0)$ has slope $1/f'(u_0(x_1))$, which is less than $1/f'(u_0(x_2))$, (that of the characteristic through $(x_2, 0)$. So these lines must cross at a point $P = (\bar{x}, \bar{t})$ with $\bar{t} > 0$ and $\bar{x} > x_2$. The solution must be discontinuous at P, since these two crossing lines would give it different values there.

(d) $\bar{t} = (x_2 - x_1)/[f'(u_0(x_1)) - f'(u_0(x_2))]$.

15. (a) Since the "rectangle" D does not touch the x axis and $\phi = 0$ on ∂D and outside D, equation (25) becomes

$$\iint [u\phi_t + f(u)\phi_x] \; dxdt = 0. \tag{i}$$

Since $(u\phi)_t + [f(u)\phi]_x = [u_t + f(u)_x]\phi + [u\phi_t + f(u)\phi_x]$, we have

$$\iint_{D_i} [u\phi_t + f(u)\phi_x] \; dxdt = \iint_{D_i} [(u\phi)_t + (f(u)\phi)_x] \; dxdt - \iint_{D_i} [u_t + f(u)_x]\phi \; dxdt.$$

But u is C^1 on the interior of D_i, and so Exercise 4(b) says $u_t + f(u)_x = 0$ there. Thus

$$\iint_{D_i} [u\phi_t + f(u)\phi_x] \; dxdt = \iint_{D_i} +[(u\phi)_t + (f(u)\phi)_x] \; dxdt. \tag{ii}$$

(b) By Green's theorem,

$$\iint_{D_i} [(f(u)\phi)_x - (-u\phi)_t] \; dxdt = \int \partial D_i[(-u\phi) \; dx + f(u)\phi \; dt],$$

and so expression (ii) becomes

$$\iint [u\phi_t + f(u)\phi_x] \; dxdt = \int_{\partial D_i} \phi[-u \; dx + f(u) \; dt]$$

Adding the above for $i = 1, 2$ and using expression (i) gives

$$0 = \int_{\partial D_i} \phi[-u \; dx + f(u) \; dt] + \int_{\partial D_2} \phi[-u \; dx + f(u) \; dt]$$

The union of these two boundaries traverses ∂D once and that portion of Γ within D in each direction, once with the values u_1 and once with the values u_2. Since $\phi = 0$ outside of D and on ∂D_2 this becomes $0 = \int_\Gamma \phi\{[-u] \; dx + [f(u)] \; dt\}$.

(c) Since $\phi = 0$ outside D, the first integral is the same as that of the second conclusion of part (b). The second integral results from parameterizing the portion of Γ by $\alpha(t) = (x(t), t), t_1 \le t \le t_2$.

(d) If $[-u]s + [f(u)] = c > 0$ at P, then we can choose a small disk B_ϵ centered at P contained in D (described above) such that $[-u](dx/dt) + [f(u)] > c/2$ on the part of Γ inside B_ϵ. Now take a slightly smaller disk $B_\gamma \subset B_\epsilon$ centered at P and pick ϕ suck that $\phi \equiv 1$ on B_γ. $0 \le \phi \le 1$ on the annulus $B_\epsilon\ B_\gamma$ and $\phi \equiv 0$ outside B_ϵ. If $\alpha(t_0) = P$, then there are t_3 and t_4 with $t_1 < t_3 < t_0 < t_4 < t_2$ and $\alpha(t) \in B_\gamma fort_3 < t < t_4$. But then

$$\int_{t_1}^{t_2} \phi \left([-u]\frac{dx}{dt} + [f(u)] \right) \, dt > \frac{c}{2}(t_4 - t_3) > 0,$$

contradicting the result of part (c). A similar argument (reversing signs) works if $c < 0$.

17. Setting $P = g(u)\phi$ and $Q = -f(u)\phi$, applying Green's theorem on rectangular R, and using the function ϕ as in Exercise 4 shows that if u is a solution of $g(u)_t + f(u)_x = 0$, then

$$\iint_{t\ge 0} [g(u)\phi_t + f(u)\phi_x] \, dxdt + \int t = 0g(u_0(x))\phi(x,0) \, dx = 0.$$

This is the appropriate analogy to equation (25), defining weak solutions of $g(u)_t + f(u)_x = 0$. Thus we want u such that

$$\iint_{t\ge 0} \left(u\phi_t + \frac{1}{2}u^2\phi_x \right) dxdt + \int_{t=0} u_0(x)\phi(x,0)dx = 0 \quad \text{(i: weak)}$$

holds for all admissible ϕ but such that

$$\iint_{t\ge 0} \left(\frac{1}{2}u^2\phi_t + \frac{1}{3}u^3\phi_x \right) \, dxdt + \int_{t=0} \frac{1}{2}u_0^2(x)\phi(x,0) \, dx = 0 \quad \text{(ii: weak)}$$

fails for some admissible ϕ. The method of Exercise 11 produces the jump condition $s[g(u)] = [f(u)]$. For (a), this is $s(u_2 - u_1) = (\frac{1}{2}u_2^2 - \frac{1}{2}u_1^2)$ or

$$s = \frac{1}{2}(u_2 + u_1). \quad \text{(i: jump)}$$

For (b), it is $s(\frac{1}{2}u_2^2 - \frac{1}{2}u_1^2) = (\frac{1}{3}u_2^3 - \frac{1}{3}u_1^3)$ or

$$s = \frac{2}{3}\frac{u_2^2 + u_1u_2 + u_1^2}{u_2 + u_1}. \quad \text{(ii: jump)}$$

If we take for $u_0(x)$ a (Heaviside) function defined by $u_0(x) = 0$ for $x < 0$ and $u_0(x) = 1$ for $x > 0$, we are led to consider the function $u(x,t) = 0$ when $t > 2x$ and $u(x,t) = 1$ when $t \leq 2x$. Thus $u_1 = 1, u_2 = 0$, and the discontinuity curve Γ is given by $t = 2x$. Thus the jump condition (i: jump)(i.e., $dx/dt = \frac{1}{2}(u_1 + u_2)$ is satisfied.

For any particular ϕ, there are numbers T and a such that $\phi(x,t) = 0$ for $x \geq a$ and $t \leq T$. Letting Ω be the region $0 \leq x \leq a$ and $0 \leq t \leq T$, condition (i: weak) becomes

$$\begin{aligned} 0 &= \iint_\Omega \left(\phi_t + \frac{1}{2}\phi_x\right)\, dxdt + \int_0^a \phi(x,0)\, dx \\ &= \int_{\partial\Omega} \left(-\phi\, dx + \frac{\phi}{2}\, dt\right) + \int_0^a \phi(x,0)\, dx \\ &= -\int_0^a \phi(x,0)\, dx + \int_0^{T/2} \left[-\phi(x,2x)(-dx) + \frac{1}{2}\phi(x,2x)(-2dx)\right] \\ &\quad + \int_0^a \phi(x,0)\, dx \end{aligned}$$

Thus (i: weak) is satisfied for every ϕ, and u is a weak solution of equation (i). However, (ii: weak) cannot be satisfied for every ϕ, since the jump condition (ii: jump) fails. Indeed, if we multiply (ii: weak) by 2 and insert u, (ii: weak) becomes

$$0 = \iint_\Omega \left(\phi_t + \frac{2}{3}\phi_x\right)\, dxdt + \int_0^a \phi(x,0)\, dx.$$

The factor $\frac{1}{2}$ has changed to $\frac{2}{3}$, and the computation above now becomes

$$0 = -\frac{1}{3}\int_0^{\pi/2} \phi(x,2x)\, dx,$$

which is certainly not satisfied for every admissible ϕ.

19. For example, write $|re^{i\theta} - z'|^2 = |re^{i\theta} - r'e^{i\theta'}|^2 = |re^{i(\theta-\theta')} - r'|^2$, use $e^{i\phi} = \cos\phi + i\sin\phi, |z|^2 = z\bar{z}$, and multiply out.

The End