Math 254H **Weekly Homework 8** **Due Apr 4, 2017**

This homework is a tutorial on limits and error analysis.

Delta-epsilon notation. We say $\lim_{x\to a} f(x) = L$, or alternatively $f(x) \to L$ as $x \to a$, when any required output error tolerance $\epsilon > 0$ can be guaranteed by some input error tolerance $\delta > 0$: that is, $|x - a| < \delta$ guarantees $|f(x) - L| < \epsilon$.

We say $f : \mathbb{R} \to \mathbb{R}$ is *continuous* at $x = a$ when $\lim_{x\to a} f(x) = f(a)$.

Example: Prove that $f(x) = 2x+1$ is continuous at $x = a$.
PROOF: We must show $\lim_{x\to a} f(x) = f(a)$. Given a required output tolerance $\epsilon > 0$ (for example $\epsilon = 0.01$), we set the input tolerance at $\delta = \frac{1}{2}\epsilon$ (which would be $\delta = 0.005$ in our example). If x meets the input tolerance $|x-a| < \delta = \frac{1}{2}\epsilon$, then the output error is $|f(x) - f(a)| = |2x+1-(2a+1)| = 2|x-a| < \epsilon$, satisfying the output tolerance.

Prob 1. Prove that a limit is a well-defined quantity if it exists: that is, if $\lim_{x\to a} f(x) = L_1$, and $\lim_{x\to a} f(x) = L_2$, then $L_1 = L_2$.
NOTE: The point here is that the complicated definition $\lim_{x\to a} f(x) = L$ could conceivably apply to two different numbers, both approached by $f(x)$. Show that $|L_1 - L_2| < \epsilon$ for every $\epsilon > 0$, which means $L_1 - L_2 = 0$.

Prob 2. Prove that if $\lim\limits_{x\to a} f(x) = L$ and $\lim\limits_{x\to a} g(x) = M$, then $\lim\limits_{x\to a} f(x)g(x) = LM$.
HINT: Relate the product error to the individual errors by writing $f(x)g(x)-LM = f(x)g(x) - Lg(x) + Lg(x) - LM$.

Similarly, we get that limits are compatible with addition, subtraction, multiplication, and division.

Example: If $g(x)$ is continuous at $x = a$, and $f(y)$ is continuous at $y = g(a)$ then $f(g(x))$ is continuous at $x = a$.
Proof: We must show $\lim_{x\to a} f(g(x)) = f(g(a))$. The continuity of $f(y)$ means that, given $\varepsilon > 0$, there is some input tolerance $\delta_1 > 0$ such that $|y-g(a)| < \delta_1$ guarantees $|f(y) - f(g(a))| < \epsilon$. Now, by the continuity of $g(x)$, there is also a $\delta_2 > 0$ such that $|x-a| < \delta_2$ guarantees $|g(x)-g(a)| < \delta_1$, which in turn guarantees $|f(g(x)) - f(g(a))| < \epsilon$. This shows the desired limit.

Little-o notation. For a function $g(h)$, we define the order class $o(g(h))$ of functions $\varepsilon(h)$ which become tiny relative to $g(h)$ as h goes to zero:

$$o(g(h)) \;=\; \{\varepsilon(h) \text{ with } \lim_{h\to 0} \tfrac{\varepsilon(h)}{g(h)} = 0, \text{ and } \varepsilon(0)=0\}.$$

We use this to indicate the magnitude of error in an approximation $f(h) \approx k(h)$:

$$f(h) \in k(h) + o(g(h)) \;\text{ means }\; f(h) = k(h) + \varepsilon(h) \;\text{ for }\; \varepsilon(h) \in o(g(h)).$$

Abusing notation, we write this as $f(x) = L + o(g(h))$, using "=" to mean "$\in$".

Example: $\lim_{x\to a} f(x) = L$ whenever $f(a{+}h) = L + o(1)$, meaning we have error $\frac{\varepsilon(h)}{1} = \varepsilon(h) = f(a{+}h){-}L \to 0$ as $h \to 0$.

Example. Geometric series. We have $\frac{1}{1-h} = 1 + h + h^2 + o(h^2)$, since the error is $\varepsilon(h) = \frac{1}{1-h} - (1{+}h{+}h^2) = \frac{1-1+h^3}{1-h}$, so $\frac{\varepsilon(h)}{h^2} = \frac{h}{1-h} \to 0$ as $h \to 0$.

Example. Prove that $o(h) + o(h) = o(h)$, meaning if $\varepsilon_1(h), \varepsilon_2(h) \in o(h)$, then $\varepsilon_1(h) + \varepsilon_2(h) \in o(h)$.

Proof: We have $\lim\limits_{h\to 0} \frac{\varepsilon_1(h)+\varepsilon_2(h)}{h} = \lim\limits_{h\to 0} \frac{\varepsilon_1(h)}{h} + \lim\limits_{h\to 0} \frac{\varepsilon_2(h)}{h} = 0 + 0 = 0.$

Similarly, if $C{\neq}0$, we have $C\, o(g(h)) = o(g(h))$; and if $g_1(h) \le g_2(h)$, we have: $o(g_1(h)) \subset o(g_2(h))$, $o(g_1(h)){+}o(g_2(h)) = o(g_2(h))$, and $o(g_1(h))o(g_2(h)) = o(g_1(h)g_2(h))$.

Prob 3. Re-do #2 in little-o notation. That is, if $f(a{+}h) = L + o(1)$ and $g(a{+}h) = M + o(1)$ as $h \to 0$, then $f(x)g(x) = LM + o(1)$.

HINT: This is less tricky than the previous method. Account for the case where L or M is zero.

Prob 4. Show $o(o(h)) \subset o(h)$. That is, if $\frac{\varepsilon_1(h)}{h}, \frac{\varepsilon_2(h)}{h} \to 0$, then $\frac{\varepsilon_1(\varepsilon_2(h))}{h} \to 0$.

HINT: Use $\frac{\varepsilon_1(\varepsilon_2(h))}{h} = \frac{\varepsilon_1(\varepsilon_2(h))}{\varepsilon_2(h)} \frac{\varepsilon_2(h)}{h}$. (Also consider when $\varepsilon_2(h) = 0$ for some $h \neq 0$.)

Derivatives. We say $f(x)$ has derivative $f'(a)$ when $f(a{+}h) = f(a){+}f'(a)h{+}o(h)$.

Prob 5. Prove that if $f'(a)$ exists, then it is unique: that is, if $f(a{+}h) = f(a) + d_1 h + o(h) = f(a) + d_2 h + o(h)$, then $d_1 = d_2$.

Prob 6. Prove that if $f'(g(a))$ and $g'(a)$ exist, then the composition $k(x) = f(g(x))$ has derivative $k'(a) = f'(g(a))\, g'(a)$.

HINT: Combine $g(a{+}h) = g(a) + g'(a)h + o(h)$ and $f(b{+}h) = f(b) + f'(b)h + o(h)$ for $b = g(a)$ and any h going to zero.