Math 254H # Weekly Homework 11 **Due Dec 11, 2019**

1. By Newton's Law of Gravitation, a point mass produces a gravitational force field pointing toward the mass, with magnitude proportional to the inverse-square distance from the mass. (The force is undefined at the mass itself.)

a. Give a formula for the gravitational field $\vec{G}(x, y, z)$ of a point mass at the origin in $\mathbb{R}^3$, assuming the constant of proportionality is 1.

b. Compute the curl of $\vec{G}$, and show it is a conservative vector field.

c. Find a potential energy function $e(x, y, z)$ with $\vec{G} = \nabla e$, normalized so that $e(x, y, z) \to 0$ as $(x, y, z) \to \infty$. *Hint:* Do not integrate from $(0, 0, 0)$, where the field is undefined.

d. Compute the flux of $\vec{G}$ out of the sphere of radius ρ centered at $(0, 0, 0)$.

e. Prove that the flux of $\vec{G}$ out of any closed surface S equals -4π if $(0, 0, 0)$ is enclosed by S, and zero if it is outside S. *Hint:* Apply the Divergence Theorem to the solid region R between S and a small sphere centered at $(0, 0, 0)$.

2. When a moth sees a light at night, it navigates so as to keep a constant angle α between its velocity vector and the direction of the light. If the light is a star, this results in a straight-line path; but if the light is a candle, the moth is constantly turning toward the light-source.

a. Assuming the candle is at $(0, 0)$ and writing the resulting path in polar form $\vec{c}(t) = f(t)(\cos(t), \sin(t))$, find a formula for the radius function $f(t)$ depending on the parameter α. *Hint:* Express the constant $\cos(\alpha)$ as a dot product, and solve the resulting easy differential equation for $f(t)$. Don't worry about initial values.

b. What happens in the end, as $t \to \infty$? (The result will depend on α.)

3. Recall that an affine mapping $A : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear mapping L shifted by a constant vector $\vec{a}$, so that $A(\vec{v}) = L(\vec{v}) + \vec{a}$.

a. Let $A = A_{\theta,\vec{c}}$ be the rotation of $\mathbb{R}^2$ around a center point $\vec{c}$ by counterclockwise angle θ. Show A is an affine mapping, and find a formula for $A(x, y)$. *Hint:* To rotate around $\vec{c}$, shift $\vec{c}$ to the origin $\vec{0}$, rotate around $\vec{0}$, then shift $\vec{0}$ back to $\vec{c}$.

b. In the geocentric model of astronomy, each planet rotated around a center point, and the center point was itself rotating around the Earth, which was fixed at the origin. The resulting path is called an *epicycloid*.

PROBLEM: Detote the path of the rotating center point by $\vec{c}(t)$, and the path of the planet by $\vec{p}(t)$. From the initial points $\vec{c}(0) = (2, 0)$ and $\vec{p}(0) = (2, 1)$, write formulas for $\vec{c}(t)$ and $\vec{p}(t)$. *Hint:* Act on $\vec{p}(0)$ by $A_{t,\vec{c}(t)}$. Check by plotting in W|A.

EXAMPLE: `parametric plot {{cos(t),-sin(t)},{sin(t),cos(t)}}.{1,1} for t=0 to pi`