Math 254H **Weekly Homework 10** **Due Nov 14, 2019**

This is a tutorial on definitions and proofs for limits and error analysis. In each proof, you may use problems and propositions which appeared earlier.

Delta-epsilon framework. Definitions:

- We say $\lim_{x\to a} f(x) = L$, or $f(x) \to L$ as $x \to a$, when any required output error tolerance $\varepsilon > 0$ can be guaranteed by some input error tolerance $\delta > 0$: that is, $0 < |x - a| < \delta$ guarantees $|f(x) - L| < \varepsilon$.

  This definition does not evaluate the limit, only rigorously verifies a given L as the limiting value. There might be *no* L satisfying the definition, in which case the limit *does not exist*.

- We say $f : \mathbb{R} \to \mathbb{R}$ is *continuous* at $x = a$ when $\lim_{x\to a} f(x) = f(a)$.

PROPOSITION: $f(x) = x^2$ is continuous at $x = 5$.

Proof: We must show $\lim_{x\to 5} x^2 = 5^2 = 25$. For any given output error tolerance $\varepsilon > 0$ (for example $\varepsilon = 0.1$), we set the input error tolerance at $\delta = \min(1, \varepsilon/11)$ ($\delta = 0.009$ in our example). Assume $|x - 5| < \delta$ meets the input tolerance, so $|x-5| < \varepsilon/11$ and $|x-5| < 1$, so $4 < x < 6$ and $|x + 5| < 11$. The output error is:

$$|x^2 - 5^2| \;=\; |(x-5)(x+5)| \;=\; |x-5|\,|x+5| \;<\; (\tfrac{\varepsilon}{11})(11) \;=\; \varepsilon.$$

Thus, a sufficiently small input error δ guarantees a given output error ε. ■

PROPOSITION: If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, then $\lim_{x\to a} f(x) + g(x) = L + M$.
Proof: Given $\varepsilon > 0$, the known limits give us $\delta_1 > 0$ such that $0 < |x - a| < \delta_1$ guarantees $|f(x) - L| < \varepsilon/2$, and $\delta_2 > 0$ such that $0 < |x - a| < \delta_2$ guarantees $|g(x) - M| < \frac{1}{2}\varepsilon$. (Here $\frac{1}{2}\varepsilon > 0$ is the given error tolerance for the known limits.) Assume $|x - a| < \delta = \min(\delta_1, \delta_2)$. Then:

$$\begin{aligned} |f(x) + g(x) - (L + M)| &= |(f(x)-L) + (g(x)-M)| \\ &\le |f(x)-L| + |g(x)-M| \\ &< \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} \;=\; \varepsilon. \end{aligned}$$

Here we used the *Triangle Inequality*: $|a + b| \le |a| + |b|$. ■

Prob 1. Prove a limit cannot converge to two different numbers: that is, if $\lim_{x\to a} f(x) = L_1$ and $\lim_{x\to a} f(x) = L_2$, then $L_1 = L_2$.

Hints: The complicated definition $\lim\limits_{x\to a} f(x) = L$ could conceivably apply to two different numbers; but show $|L_1 - L_2| < \varepsilon$ for every $\varepsilon > 0$, so $L_1 - L_2 = 0$.

Prob 2. Prove if $\lim\limits_{x\to a} f(x) = L$, $\lim\limits_{x\to a} g(x) = M$, then $\lim\limits_{x\to a} f(x)g(x) = LM$.
Hint: Relate error in the product to the errors in each factor by writing:

$$f(x)g(x) - LM \;=\; f(x)g(x) - L\,g(x) + L\,g(x) - LM.$$

Similarly, limits are compatible with addition, subtraction, multiplication, division. Also with composition (substitution), if the functions are continuous:

PROPOSITION: If $g(x)$ is continuous at $x = a$, and $f(y)$ is continuous at $y = g(a)$ then $f(g(x))$ is continuous at $x = a$.

Proof: We must show $\lim_{x\to a} f(g(x)) = f(g(a))$. The continuity of $f(y)$ means that, given $\varepsilon > 0$, there is some input error $\delta' > 0$ such that $|y - g(a)| < \delta'$ guarantees $|f(y) - f(g(a))| < \varepsilon$. Now, by the continuity of $g(x)$, we can take $\delta' > 0$ as the output error for $g(x)$, and find a $\delta > 0$ such that $|x-a| < \delta$ guarantees $|g(x) - g(a)| < \delta'$, which in turn guarantees $|f(g(x)) - f(g(a))| < \varepsilon$. This shows the desired limit.

Little-o notation. For a magnitude function $M(h)$, we define order class $o(M(h))$ as all functions $\varepsilon(h)$ which become tiny relative to $M(h)$ as h approaches zero:

$$o(M(h)) \;=\; \left\{ \varepsilon(h) \text{ with } \lim_{h\to 0} \frac{|\varepsilon(h)|}{|M(h)|} = 0 \text{ and } \varepsilon(0) = 0 \right\}.$$

This measures the error in an approximation $f(h) \approx k(h)$ for small $h \approx 0$:

$$f(h) = k(h) + o(M(h)) \;\text{ means }\; f(h) = k(h) + \varepsilon(h) \;\text{ for some } \varepsilon(h) \in o(M(h)).$$

In more conventional terminology, $f(h)$ is an *element* of the shifted set:

$$f(h) \;\in\; k(h) + o(M(h)) = \{k(h) + \varepsilon(h) \text{ for } \varepsilon(h) \in o(M(h))\}.$$

PROPOSITION: $\lim_{x\to a} f(x) = f(a)$ is equivalent to $f(a{+}h) = f(a) + o(1)$.

Proof: By the Sum of Limits Theorem, we have the equivalences:

$$\lim_{x\to a} f(x) = f(a) \iff \lim_{x\to a} f(x) - f(a) = 0 \iff \lim_{x\to a} \varepsilon(x{-}a) = 0,$$

where $\varepsilon(h) = f(a{+}h){-}f(a)$. Substituting $h = x{-}a$, this is equivalent to $\lim\limits_{h\to 0} \varepsilon(h) = \varepsilon(0) = 0$, meaning $\varepsilon(h) \in o(1)$, or $f(a{+}h) = f(a) + \varepsilon(h) = f(a) + o(1)$. ∎

PROPOSITION. Letting $o(h) + o(h) = \{\varepsilon_1(h) + \varepsilon_2(h) \text{ for } \varepsilon_1(h), \varepsilon_2(h) \in o(h)\}$, we have $o(h) + o(h) = o(h)$.

Proof: Since $\varepsilon(h) = 0 \in o(h)$, clearly $o(h) \subset o(h){+}o(h)$. For the opposite inclusion, take $\varepsilon_1(h), \varepsilon_2(h) \in o(h)$, and compute:

$$\lim_{h\to 0} \frac{\varepsilon_1(h) + \varepsilon_2(h)}{h} \;=\; \lim_{h\to 0} \frac{\varepsilon_1(h)}{h} + \lim_{h\to 0} \frac{\varepsilon_2(h)}{h} \;=\; 0 + 0 \;=\; 0.$$

Thus $\varepsilon_1(h) + \varepsilon_2(h) \in o(h)$ and $o(h) + o(h) \subset o(h)$. ∎

Similarly, for $C{\neq}0$, we have $C\, o(M(h)) = o(M(h))$. If $|M_1(h)| \le |M_2(h)|$, then:

$$\begin{aligned} o(M_1(h)) &\subset o(M_2(h)) \\ o(M_1(h)) + o(M_2(h)) &= o(M_2(h)) \\ o(M_1(h))\, o(M_2(h)) &= M_1(h)\, o(M_2(h)) \;=\; o(M_1(h) M_2(h)). \end{aligned}$$

PROP: Letting $o(o(M(h))) = \bigcup_{\varepsilon(h)\in o(M(h))} o(\varepsilon(h))$, we have $o(o(M(h))) = o(M(h))$.

Proof: Clearly $o(o(h)) \subset o(h)$. For for the opposite inclusion, we must show that for any $\varepsilon_1(h) \in o(M(h))$, there is some $\varepsilon_2(h) \in o(M(h))$ with $\varepsilon_1(h) \in o(\varepsilon_2(h))$. By definition, we have the ratio $\rho(h) = |\varepsilon_1(h)/M(h)| \to 0$ as $h \to 0$, so also $\sqrt{\rho(h)} \to 0$. Thus $\varepsilon_2(h) = \sqrt{\rho(h)} M(h) \in o(M(h))$, and we have:

$$\left|\frac{\varepsilon_1(h)}{\varepsilon_2(h)}\right| \;=\; \frac{\rho(h)\,|M(h)|}{\sqrt{\rho(h)}\,|M(h)|} \;=\; \sqrt{\rho(h)} \;\to\; 0,$$

so $\varepsilon_1(h) \in o(\varepsilon_2(h))$, and we conclude $o(h) \subset o(o(h))$.

Prob 3. Re-do #2 in little-o notation, for continuous functions: if $f(a{+}h) = f(a) + o(1)$ and $g(a{+}h) = g(a) + o(1)$ as $h \to 0$, then $f(x)g(x) = f(a)g(a) + o(1)$.

Hint: This is immediate, using the above facts. Also consider if $f(a)$ or $g(a) = 0$.

Prob 4. For two classes of functions $c_1(h), c_2(h)$, define their composition:

$$c_1(h) \circ c_2(h) = \{\varepsilon_1(\varepsilon_2(h)) \text{ for } \varepsilon_1(h) \in c_1(h), \varepsilon_2(h) \in c_2(h)\}.$$

Show that $o(h) \circ (Ch + o(h)) \subset o(h)$ for any constant C: that is, if $\varepsilon_1(h), \varepsilon_2(h) \in o(h)$, then $\varepsilon_1(Ch + \varepsilon_2(h)) \in o(h)$. *Extra Credit:* Show $o(h) \circ (Ch + o(h)) = o(h)$.
HINT: Use $\frac{\varepsilon_1(Ch+\varepsilon_2(h))}{h} = \frac{\varepsilon_1(Ch+\varepsilon_2(h))}{Ch+\varepsilon_2(h)} \frac{Ch+\varepsilon_2(h)}{h}$. (What to do if $Ch + \varepsilon_2(h) = 0$?)

Derivatives. A *derivative* means a limit $f'(a) = \lim\limits_{x \to a} \frac{f(x)-f(a)}{x-a}$, if it exists.

In o-notation, the derivative is the slope in a good linear approximation $f(a+h) = f(a) + f'(a)h + o(h)$.

Prob 5. Prove that if a good linear approximation exists, then it is unique:

$$f(a+h) \;=\; f(a) + m_1 h + o(h) \;=\; f(a) + m_2 h + o(h) \quad \Rightarrow \quad m_1 = m_2.$$

PROPOSITION: $f(a+h) = f(a) + mh + o(h)$ if and only if $m = f'(a)$.

Proof. Suppose $f(a+h) = f(a) + mh + o(h)$, meaning $f(a+h) = f(a) + mh + \varepsilon(h)$ for a function $\varepsilon(h) \in o(h)$, so that $\lim\limits_{h \to 0} \frac{\varepsilon(h)}{h} = 0$. Solving for m and letting $h \to 0$:

$$m \;=\; \lim_{h \to 0} \frac{f(a+h)-f(a)}{h} + \frac{\varepsilon(h)}{h} \;=\; f'(a) + 0.$$

Conversely, if $m = f'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$, then $\frac{f(a+h)-f(a)}{h} = m + o(1)$ by a previous proposition, so $f(a+h) - f(a) = mh + h\, o(1) = mh + o(h)$ and $f(a+h) = f(a) + mh + o(h)$. ∎

Prob 6. Prove that if $f'(g(a))$ and $g'(a)$ exist, then the composition $k(x) = f(g(x))$ has derivative $k'(a) = f'(g(a))\, g'(a)$.

Higher-order approximation.

Prob 7. Prove the geometric series approximation $\frac{1}{1-h} \;=\; 1 + h + h^2 + o(h^2)$.

Quotient Rule.

PROPOSITION: If $\lim_{h\to 0} q(h) = 0$, then $\frac{1}{1-q(h)} = 1 + q(h) + o(q(h))$.
Proof: The error in the approxmation is:

$$\varepsilon(h) \;=\; \frac{1}{1-q(h)} - (1{+}q(h)) \;=\; \frac{1-(1{-}q(h)^2)}{1-q(h)} \;=\; \frac{q(h)^2}{1-q(h)},$$

so $\varepsilon(h)/q(h) = q(h)/(1{-}q(h)) \to 0/(1{-}0) = 0$. ∎

Finally, we approximate $\frac{f(x)}{g(x)}$ near $x=a$, assuming $g(a)\neq 0$:

$$\begin{aligned}
\frac{f(a{+}h)}{g(a{+}h)} &= \frac{f(a)+f'(a)h+o(h)}{g(a)+g'(a)h+o(h)} \\
&= \frac{f(a)+f'(a)h+o(h)}{g(a)(1-q(h))}, \quad q(h) = -\tfrac{g'(a)}{g(a)}h - o(h) \\
&= \tfrac{1}{g(a)}\Big(f(a)+f'(a)h+o(h)\Big)\Big(1+q(h)+o(q(h))\Big) \\
&= \tfrac{1}{g(a)}\Big(f(a)+f'(a)h+o(h)\Big)\Big(1-\tfrac{g'(a)}{g(a)}h+o(h)\Big) \\
&= \tfrac{1}{g(a)}\Big(f(a)+f'(a)h-f(a)\tfrac{g'(a)}{g(a)}h+o(h)\Big).
\end{aligned}$$

The coefficient of h gives the derivative of $\frac{f(x)}{g(x)}$ at $x=a$:

$$\frac{f'(a)-f(a)\frac{g'(a)}{g(a)}}{g(a)} \;=\; \frac{f'(a)g(a)-f(a)g'(a)}{g(a)^2}.$$