1a. The Jacobian derivatives of $f(x,y)$ and $P(r,\theta) = (r\cos(\theta), r\sin(\theta))$ are:

$$[Df_{(x,y)}] = \nabla f(x,y) = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right], \qquad [DP_{(r,\theta)}] = \begin{bmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{bmatrix}.$$

By the Chain Rule, the derivative of $g(r,\theta) = f(P(r,\theta))$ is the product matrix:

$$
\begin{aligned}
[Dg_{(r,\theta)}] &= \nabla g(r,\theta) = \left[\frac{\partial f}{\partial x}(P(r,\theta)), \frac{\partial f}{\partial y}(P(r,\theta))\right] \cdot \begin{bmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{bmatrix}, \\
&= \left[\frac{\partial f}{\partial x}\cos(\theta) + \frac{\partial f}{\partial y}\sin(\theta) \;,\; -\frac{\partial f}{\partial x}r\sin(\theta) + \frac{\partial f}{\partial y}r\cos(\theta)\right]
\end{aligned}
$$

where $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ are taken at $P(r,\theta)$.

1b. We get the same result in Leibnitz notation, writing $z = f(x,y)$, $x = r\cos(\theta)$, $y = r\sin(\theta)$ and computing:

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial f}{\partial x}(x,y)\cos(\theta) + \frac{\partial f}{\partial y}(x,y)\sin(\theta),$$

and similarly for $\frac{\partial z}{\partial \theta}$.

2. For $\mathbf{c}(\theta) = P(r(\theta), \theta)$, we have:

$$
\begin{aligned}
\mathbf{c}'(\theta) &= [DP_{(r(\theta),\theta)}] \cdot [D(r(\theta),\theta))_\theta] \\[2mm]
&= \begin{bmatrix} \cos(\theta) & -r(\theta)\sin(\theta) \\ \sin(\theta) & r(\theta)\cos(\theta) \end{bmatrix} \cdot \begin{bmatrix} r'(\theta) \\ 1 \end{bmatrix} \\[2mm]
&= \begin{bmatrix} \cos(\theta)r'(\theta) - r(\theta)\sin(\theta) \\ \sin(\theta)r'(\theta) + r(\theta)\cos(\theta) \end{bmatrix} \\[2mm]
&= r'(\theta)\begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix} + r(\theta)\begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}'
\end{aligned}
$$

The last expression agrees with the Product Rule applied to $\mathbf{c}(\theta) = r(\theta)(\cos(\theta), \sin(\theta))$.

3. For $m(x,y) = xy$ and $F(x,y) = (f(x,y), g(x,y))$, the Jacobians are:

$$[Dm_{x,y}] = \nabla m(x,y) = [y \;\; x], \qquad [DF_{(x,y)}] = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}.$$

Letting $F(a,b) = (f(a,b), g(a,b)) = (f,g)$, the Chain Rule says:

$$
\begin{aligned}
\nabla(fg)(a,b) &= [D(m\circ F)_{(a,b)}] = [Dm_{F(a,b)}] \cdot [DF_{(a,b)}] \\[2mm]
&= \nabla m(f,g) \cdot \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix} = [g \;\; f] \cdot \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix} \\[2mm]
&= \left[g\frac{\partial f}{\partial x} + f\frac{\partial g}{\partial x} \;,\; g\frac{\partial f}{\partial y} + f\frac{\partial g}{\partial y}\right] \\[2mm]
&= g\left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right] + f\left[\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}\right] \\[2mm]
&= g(a,b)\nabla f(a,b) + f(a,b)\nabla g(a,b).
\end{aligned}
$$

4. Given $h(x, y) = f(x, g(x, y))$, we define $G(x, y) = (x, g(x, y))$ and compute:

$$\nabla h(a, b) \;=\; [D(f \circ G)_{(a,b)}] \;=\; [Df_{G(a,b)}] \cdot [DG_{(a,b)}]$$

$$=\; \nabla f(a, g(a, b)) \cdot \begin{bmatrix} 1 & 0 \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}$$

$$=\; \left[\frac{\partial f}{\partial x}(a, g(a, b)) + \frac{\partial f}{\partial y}(a, g(a, b)) \frac{\partial g}{\partial x}(a, b) \;,\; \frac{\partial f}{\partial y}(a, g(a, b)) \frac{\partial g}{\partial y}(a, b) \right]$$

Notice that $\frac{\partial f}{\partial y}$ means the partial derivative of $f(x, y)$ with respect to the second variable, *before* substituting $g(x, y)$ for that variable.

This might be clearer in Leibnitz letter-notation:

$$z = f(x, y), \quad x = u, \quad y = g(u, v),$$

$$\frac{\partial z}{\partial u} \;=\; \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \;=\; \frac{\partial f}{\partial x}(u, g(u, v)) + \frac{\partial f}{\partial y}(u, g(u, v)) \frac{\partial g}{\partial u}(u, v),$$

$$\frac{\partial z}{\partial v} \;=\; \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} \;=\; 0 + \frac{\partial f}{\partial y}(u, g(u, v)) \frac{\partial g}{\partial v}(u, v),$$