Math 254H **Homework 2 Solutions** **Spr 2017**

1. Proposition: If $L_1, L_2 : \mathbb{R}^2 \to \mathbb{R}^2$ are linear mappings, then so is the composition $L_3 = L_1 \circ L_2$.

Proof. First, it is clear that the composition $L_3(\mathbf{v}) = L_1(L_2(\mathbf{v}))$ is a well-defined mapping, taking $L_3 : \mathbb{R}^2 \xrightarrow{L_2} \mathbb{R}^2 \xrightarrow{L_1} R^2$. We must also check the defining properties of a linear mapping, assuming these properties for L_1, L_2:

$$\begin{aligned} L_3(\mathbf{u}+\mathbf{v}) &= L_1(L_2(\mathbf{u}+\mathbf{v})) \;=\; L_1(L_2(\mathbf{u}) + L_2(\mathbf{v})) \\ &= L_1(L_2(\mathbf{u})) + L_1(L_2(\mathbf{v})) \;=\; L_3(\mathbf{u}) + L_3(\mathbf{v}), \end{aligned}$$

$$L_3(s\mathbf{v}) \;=\; L_1(L_2(s\mathbf{v})) \;=\; L_1(sL_2(\mathbf{v})) \;=\; sL_1(L_2(\mathbf{v})) \;=\; sL_3(\mathbf{v}).$$

Hence L_3 is linear.

2a. Given $P(\mathbf{v}) = \frac{\mathbf{v}\cdot\mathbf{a}}{\mathbf{a}\cdot\mathbf{a}}\,\mathbf{a}$, we have:

$$P(P(\mathbf{v}) \;=\; \frac{\left(\frac{\mathbf{v}\cdot\mathbf{a}}{\mathbf{a}\cdot\mathbf{a}}\mathbf{a}\right)\cdot\mathbf{a}}{\mathbf{a}\cdot\mathbf{a}}\,\mathbf{a} \;=\; \frac{\mathbf{v}\cdot\mathbf{a}}{\mathbf{a}\cdot\mathbf{a}}\,\frac{\mathbf{a}\cdot\mathbf{a}}{\mathbf{a}\cdot\mathbf{a}}\,\mathbf{a} \;=\; P(\mathbf{v}).$$

Thus $\mathrm{Proj}_{\mathbf{a}} \circ \mathrm{Proj}_{\mathbf{a}} = \mathrm{Proj}_{\mathbf{a}}$, and it is geometrically clear that projecting a second time to the **a**-line has no effect. If $\mathbf{a} = (a_1, a_2)$, the matrix is:

$$[\mathrm{Proj}_{\mathbf{a}}] \;=\; \tfrac{1}{a_1^2+a_2^2}\begin{bmatrix} a_1^2 & a_1a_2 \\ a_1a_2 & a_2^2 \end{bmatrix}$$

where the scalar at left multiplies every entry of the matrix.

2b. Given $R(\mathbf{v}) = \mathbf{v} - 2\frac{\mathbf{v}\cdot\mathbf{a}}{\mathbf{a}\cdot\mathbf{a}}\,\mathbf{a}$, we have:

$$\begin{aligned} R(R(\mathbf{v})) &= \left(\mathbf{v} - 2\tfrac{\mathbf{v}\cdot\mathbf{a}}{\mathbf{a}\cdot\mathbf{a}}\mathbf{a}\right) - 2\frac{\left(\mathbf{v}-2\frac{\mathbf{v}\cdot\mathbf{a}}{\mathbf{a}\cdot\mathbf{a}}\mathbf{a}\right)\cdot\mathbf{a}}{\mathbf{a}\cdot\mathbf{a}}\,\mathbf{a} \\ &= \mathbf{v} - 2\tfrac{\mathbf{v}\cdot\mathbf{a}}{\mathbf{a}\cdot\mathbf{a}}\,\mathbf{a} - 2\tfrac{\mathbf{v}\cdot\mathbf{a}}{\mathbf{a}\cdot\mathbf{a}}\,\mathbf{a} + 4\tfrac{\mathbf{v}\cdot\mathbf{a}}{\mathbf{a}\cdot\mathbf{a}}\,\mathbf{a} \\ &= \mathbf{v}. \end{aligned}$$

That is, $\mathrm{Ref}_{\mathbf{a}} \circ \mathrm{Ref}_{\mathbf{a}} = \mathrm{I}$, the identity mapping $I(\mathbf{v}) = \mathbf{v}$ with matrix $[\,\mathrm{I}\,] = \left[\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right]$.

2c. The rotation is defined by: $R(x, y) = x\,(\cos\theta, \sin\theta) + y(-\sin\theta, \cos\theta)$. In particular taking $(x, y) = R(\mathbf{i}) = (\cos\theta, \sin\theta)$ gives:

$$\begin{aligned} R(R(\mathbf{i})) &= R(\cos\theta, \sin\theta) = \cos\theta(\cos\theta, \sin\theta) + \sin\theta(-\sin\theta, \cos\theta) \\ &= (\cos^2\theta - \sin^2\theta\,,\, 2\sin\theta\cos\theta), \end{aligned}$$

and similarly $R(R(\mathbf{j})) = (-2\sin\theta\cos\theta\,, \cos^2\theta - \sin^2\theta)$. Simplifying with the Double Angle Formulas from trigonometry, we find the matrix:

$$[\mathrm{Rot}_\theta \circ \mathrm{Rot}_\theta] \;=\; \left[\begin{array}{c|c} \cos(2\theta) & -\sin(2\theta) \\ \sin(2\theta) & \cos(2\theta) \end{array}\right] \;=\; [\,\mathrm{Rot}_{2\theta}\,].$$

Geometrically, this means two successive rotations by θ produce rotation by 2θ.

2d. If $R_1(x,y) = x(0,1)+y(-1,0) = (-y,x)$ and $R_2(x,y) = (-x,y)$, then: $R_1(R_2(x,y)) = R_2(-x,y) = (-y,-x)$, with matrix: $[\,R_1 \circ R_2\,] = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$. Drawing this shows it is $\mathrm{Ref}_{(1,1)}$, reflection of the direction $\mathbf{a} = (1,1)$ across the perpendicular line $y = -x$.

3. The linear mapping $R = \mathrm{Ref}_{\mathbf{a}}$ reflects the vector $\mathbf{a} = (a_1, a_2)$ across its perpendicular line $a_1x + a_2y = 0$. Since we know $R(\mathbf{v}) = \mathbf{v} - 2\frac{\mathbf{v}\cdot\mathbf{a}}{\mathbf{a}\cdot\mathbf{a}}\,\mathbf{a}$, we have:

$$R(\mathbf{i}) \;=\; (1,0) - 2\tfrac{a_1}{a_1^2+a_2^2}(a_1,a_2) \;=\; \tfrac{1}{a_1^2+a_2^2}(a_2^2-a_1^2\,,-2a_1a_2)$$

and similarly $R(\mathbf{j}) = \frac{1}{a_1^2+a_2^2}(-2a_1a_2\,, a_1^2-a_2^2)$. Writing these as columns gives the matrix:

$$[\,\mathrm{Ref}_{\mathbf{a}}\,] \;=\; \tfrac{1}{a_1^2+a_2^2}\left[\begin{array}{c|c} a_2^2-a_1^2 & -2a_1a_2 \\ -2a_1a_2 & a_1^2-a_2^2 \end{array}\right].$$

The scalar on the left multiplies each element of the matrix; it disappears if $|\mathbf{a}| = 1$.