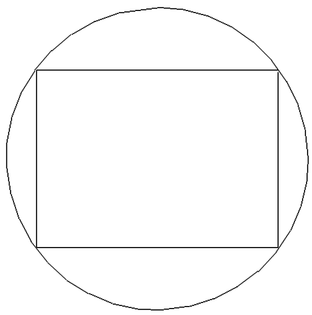
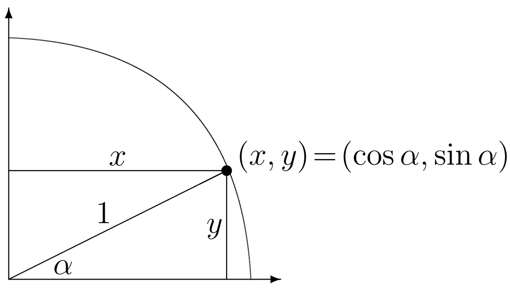
Math 254H **Weekly Homework 3** **Spring 2017**

Optimizing a Sawmill

We seek the most efficient ways to cut up round logs into rectangular planks and beams. We look at the cross-section of our log, which we assume to be a circle of radius 1 foot.

To begin, we want to find the dimensions of a single beam of maximum cross-section area cut out of this circle. It is convenient to look at a quadrant.

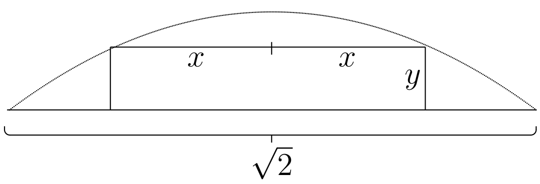




1. Consider the variable α, the angle between the x-axis and the corner point. Then the corner point has coordinates $(\cos\alpha, \sin\alpha)$, and we want to maximize the area function $f(\alpha) = \cos(\alpha)\sin(\alpha)$ on the interval $0 \leq \alpha \leq \pi/2$. (Here $f(\alpha)$ is $\frac{1}{4}$ of the beam's cross-section area.)

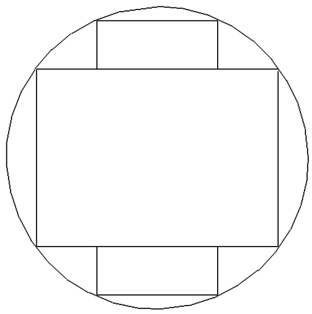
Solve this problem algebraically, showing that the optimal beam is *square*. In your computations, keep in mind the identities $\sin(2\theta) = 2\cos\theta\sin\theta$, $\cos(2\theta) = \cos^2\theta - \sin^2\theta$. What is special about the values of f at the boundary points $\alpha = 0, \pi/2$?

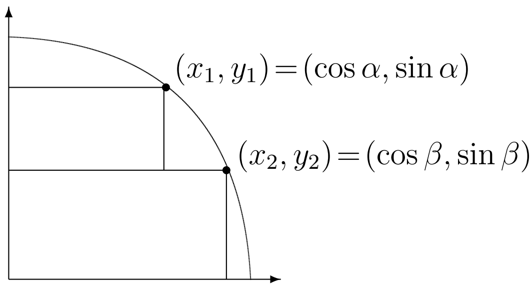
2. Now suppose we have cut out the square beam described above, and we want to cut extra planks of maximum cross-section out of the remaining pieces.



That is, we maximize the area xy subject to the relation $x^2 + (y + \frac{\sqrt{2}}{2})^2 = 1$. Explain these equations, and find the maximum of the area function. (Do you need one- or two-variable calculus?)

3. Next consider simultaneously cutting out a thick beam and two planks above and below it.



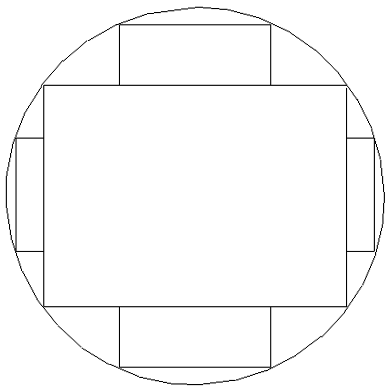


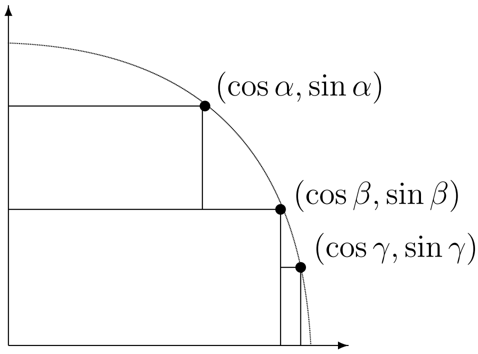
The total area cut out of the quadrant is: $x_1y_1 + x_2y_2 - x_1y_2$. (Explain!) Write the area as a function $f(\alpha, \beta)$ of the angles α, β marked on the circle, and maximize over the region $0 \le \alpha, \beta \le \pi/2$.

Include a printout of the contour plot of $f(\alpha, \beta)$ over this region, and approximate the maximum point or points to two decimal places.

Also: is the result the same as successively cutting out a maximal beam and then maximal planks (as in Probs 1,2)? Doesn't it *have* to be the same? Explain, referring to the contour plot.

4. Now consider a beam and four surrounding planks:





Write down the combined area of the wood cut out of the quadrant at right, as a function $f(\alpha, \beta, \gamma)$. Be careful to count each region only once, subtracting out overlaps as in Prob 3.

5. Maximize the three-variable area function from Prob 4. Here, as usually with practical problems, the only feasible method is the numerical gradient-flow technique.

Start with a rough approximation (really a guess) for the maximum point, $\mathbf{v}_0 = (\alpha_0, \beta_0, \gamma_0)$, and compute $\nabla f(\mathbf{v}_0)$. If $\mathbf{v}_0$ were really the maximum, we would have $\nabla f(\mathbf{v}_0) = (0, 0, 0)$; but since it is only an estimate, the non-zero gradient points in the direction of maximum increase of $f(\mathbf{v})$. Obtain an improved estimate $\mathbf{v}_1 = \mathbf{v}_0 + \epsilon \nabla f(\mathbf{v}_0)$, where ϵ is a small fixed parameter, such as $\epsilon = 0.5$.

Iterate this process to get a sequence of approximations $\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \ldots$, converging to the true max point. If the algorithm takes too large steps, it might oscillate back and forth around the maximum: if so, decrease ϵ. If it takes too small steps, it might converge very slowly: increase ϵ. A more serious problem is that it might find one max point near the initial estimate, and miss others. Try a grid of initial values to see that they all converge to the same point.

Hand in a spreadsheet or computer algebra printout giving the maximum point correct to two decimal places.

6. Some theoretical questions about the gradient flow algorithm

a. How will the algorithm behave if there is no maximum point, and $f(\mathbf{v})$ keeps increasing in some direction?

b. How would you modify the algorithm to find a min point?

c. How would the algorithm behave close to a saddle point?