2a. Consider functions the form:

$$\mathbf{f}:\mathbb{R}^2\to\mathbb{R}^2 \qquad \mathbf{c}:\mathbb{R}\to\mathbb{R}^2 \qquad g:\mathbb{R}^2\to\mathbb{R}$$
$$\mathbf{f}(x,y)=(f_1(x,y),f_2(x,y)) \qquad \mathbf{c}(t)=(c_1(t),c_2(t)) \qquad g(x,y).$$

The Jacobian matrices of partial derivatives at $\mathbf{v}=\mathbf{a}$ or $t=a$ are:

$$[D\mathbf{f}_{\mathbf{a}}] \;=\; \left[\begin{array}{c|c} \frac{\partial \mathbf{f}}{\partial x} & \frac{\partial \mathbf{f}}{\partial y}\end{array}\right] \;=\; \left[\begin{array}{c}\nabla f_1(\mathbf{a}) \\ \hline \nabla f_2(\mathbf{a})\end{array}\right] \;=\; \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y}\end{bmatrix}$$

$$[D\mathbf{c}_a] \;=\; \left[\, \frac{d\mathbf{c}}{dt} \,\right] \;=\; [\; \mathbf{c}'(a) \;] \;=\; \begin{bmatrix} \frac{dc_1}{dt} \\ \frac{dc_2}{dt}\end{bmatrix}$$

$$[Dg_{\mathbf{a}}] \;=\; [\; \nabla g(\mathbf{a}) \;] \;=\; \begin{bmatrix} \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y}\end{bmatrix}.$$

Each of these represents a linear function which approximates the original function near a specific point $\mathbf{v}=\mathbf{a}$ or $t=a$, and it is a matrix of *numbers*, not of functions.

We form all possible compositions between these functions, and write out the matrix form of the Chain Rule.

- $g\circ\mathbf{c}:\mathbb{R}\xrightarrow{\mathbf{c}}\mathbb{R}^2\xrightarrow{g}\mathbb{R}$. The most important case of multivariable Chain Rule.

$$\begin{aligned} & [\,D(g\circ\mathbf{c})_a] && = \left[\, \tfrac{d}{dt}(g(\mathbf{c}(t))\,\right] \\ = \;& [\,Dg_{\mathbf{c}(a)}]\cdot[D\mathbf{c}_a] && = [\;\nabla g(\mathbf{c}(a))\cdot\mathbf{c}'(a)\;] \\ = \;& \begin{bmatrix} \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y}\end{bmatrix}\cdot\begin{bmatrix}\frac{dc_1}{dt} \\ \frac{c_2}{dt}\end{bmatrix} && = \begin{bmatrix} \frac{\partial g}{\partial x}\frac{\partial c_1}{\partial t}+\frac{\partial g}{\partial y}\frac{\partial c_2}{\partial t}\end{bmatrix}. \end{aligned}$$

We usually write this without matrices, and denoting $\mathbf{c}(t)=(x(t),y(t))$:

$$\frac{d}{dt}(g(\mathbf{c}(t)) \;=\; \nabla g(\mathbf{c}(t))\cdot\mathbf{c}'(t) \;=\; \frac{\partial g}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial g}{\partial y}\frac{\partial y}{\partial t}\,,$$

where it is understood that we plug in $t=a$, $x=x(a)$, $y=y(a)$.

- $\mathbf{c} \circ g : \mathbb{R}^2 \xrightarrow{g} \mathbb{R} \xrightarrow{\mathbf{c}} \mathbb{R}^2$.

$$[\,D(\mathbf{c} \circ g)_{\mathbf{a}}] \quad = \quad \begin{bmatrix} \frac{\partial}{\partial x} c_1(g(x,y)) & \frac{\partial}{\partial y} c_1(g(x,y)) \\ \frac{\partial}{\partial x} c_2(g(x,y)) & \frac{\partial}{\partial y} c_2(g(x,y)) \end{bmatrix}$$

$$= \quad [\,D\mathbf{c}_{g(\mathbf{a})}] \cdot [Dg_{\mathbf{a}}] \quad = \quad [\ \mathbf{c}'(g(\mathbf{a})\ ] \cdot [\ \nabla g(\mathbf{a})\ ]$$

$$= \begin{bmatrix} \frac{dc_1}{dt} \\ \frac{dc_2}{dt} \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix} \quad = \quad \begin{bmatrix} \frac{dc_1}{dt}\frac{\partial g}{\partial x} & \frac{dc_1}{dt}\frac{\partial g}{\partial x} \\ \frac{dc_2}{dt}\frac{\partial g}{\partial x} & \frac{dc_2}{dt}\frac{\partial g}{\partial x} \end{bmatrix}.$$

- $\mathbf{f} \circ \mathbf{c} : \mathbb{R} \xrightarrow{\mathbf{c}} \mathbb{R}^2 \xrightarrow{\mathbf{f}} \mathbb{R}^2$.

$$[\,D(\mathbf{f} \circ \mathbf{c})_a] \quad = \quad \begin{bmatrix} \frac{d}{dt} f_1(\mathbf{c}(t)) \\ \frac{d}{dt} f_2(\mathbf{c}(t)) \end{bmatrix}$$

$$= \quad [\,D\mathbf{f}_{\mathbf{c}(a)}] \cdot [D\mathbf{c}_a] \quad = \quad \begin{bmatrix} \nabla f_1(\mathbf{c}(a)) \cdot \mathbf{c}'(a) \\ \nabla f_2(\mathbf{c}(a)) \cdot \mathbf{c}'(a) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix} \cdot \begin{bmatrix} \frac{dc_1}{dt} \\ \frac{dc_2}{dt} \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x}\frac{dc_1}{dt} + \frac{\partial f_1}{\partial y}\frac{dc_2}{dt} \\ \frac{\partial f_2}{\partial x}\frac{dc_1}{dt} + \frac{\partial f_2}{\partial y}\frac{dc_2}{dt} \end{bmatrix}.$$

- $g \circ \mathbf{f} : \mathbb{R}^2 \xrightarrow{\mathbf{f}} \mathbb{R}^2 \xrightarrow{g} \mathbb{R}$.

$$[\,D(g \circ \mathbf{f})_{\mathbf{a}}] \quad = \quad \begin{bmatrix} \frac{\partial}{\partial x} g(\mathbf{f}(x,y)) & \frac{\partial}{\partial y} g(\mathbf{f}(x,y)) \end{bmatrix}$$

$$= \quad [\,Dg_{\mathbf{f}(a)}] \cdot [D\mathbf{f}_{\mathbf{a}}] \quad = \quad \begin{bmatrix} \nabla g(\mathbf{f}(a)) \cdot \frac{\partial \mathbf{f}}{\partial x} & \nabla g(\mathbf{f}(a)) \cdot \frac{\partial \mathbf{f}}{\partial y} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{\partial g}{\partial x}\frac{\partial f_1}{\partial t} + \frac{\partial g}{\partial y}\frac{\partial f_2}{\partial t} & \frac{\partial g}{\partial x}\frac{\partial f_1}{\partial t} + \frac{\partial g}{\partial y}\frac{\partial f_2}{\partial t} \end{bmatrix}.$$

2b. We illustrate $\mathbf{f}(\mathbf{c}(t))$ for the case: $\mathbf{f}(x,y) = (xe^{x^2+y^2}, ye^{x^2+y^2})$, $\mathbf{c}(t) = (t\cos(t), t\sin(t))$, and $a = \pi$. Then: $\mathbf{c}(a) = (\pi, 0)$, $\mathbf{c}'(a) = (\cos(t) - t\sin(t), \sin(t) + t\cos(t))|_{t=\pi} = (1, \pi)$, and:

$$[\,D\mathbf{f}_{\mathbf{c}(a)}\,] = \begin{bmatrix} (2x^2+1)e^{x^2+y^2} & 2xye^{x^2+y^2} \\ 2xye^{x^2+y^2} & (2y^2+1)e^{x^2+y^2} \end{bmatrix}\Bigg|_{(x,y)=(\pi,0)} = \begin{bmatrix} (2\pi^2+1)e^{\pi^2} & 0 \\ 0 & e^{\pi^2} \end{bmatrix}$$

Thus:

$$[\ D(\mathbf{f} \circ \mathbf{c})(a)\ ] = \begin{bmatrix} (2\pi^2+1)e^{\pi^2} & 0 \\ 0 & e^{\pi^2} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ \pi \end{bmatrix} = \begin{bmatrix} (2\pi^2+1)e^{\pi^2} \\ \pi e^{\pi^2} \end{bmatrix}$$