Math 254H **Thales' Theorem** **Jan 15, 2016**

PROPOSITION: In a half-circle, inscribe a triangle with one side equal to the diameter; then the opposite angle must be a right angle.

PROOF: Let A, B, C be the vertices of the triangle, O the center of the circle, and line segment $\overline{AOB}$ the diameter. Define the radius vector $\vec{v} = \vec{OA}$, with opposite radius $-\vec{v} = \vec{OB}$, and $\vec{u} = \vec{OC}$. Since the triangle $\triangle ABC$ is inscribed in the circle, these vectors are all radii with the same length: $|\vec{v}| = |\vec{u}|$.

The angle $\angle C$ consists of rays $\vec{CA} = \vec{v} - \vec{u}$ and $\vec{CB} = -\vec{v} - \vec{u} = -(\vec{v} + \vec{u})$, with dot product:

$$\begin{aligned} -(\vec{v}+\vec{u}) \cdot (\vec{v}-\vec{u}) &= -(\vec{v}\cdot\vec{v} - \vec{v}\cdot\vec{u} + \vec{u}\cdot\vec{v} - \vec{u}\cdot\vec{u}) \\ &= -|v|^2 + \vec{u}\cdot\vec{v} - \vec{u}\cdot v + |\vec{u}|^2 = 0, \end{aligned}$$

where we use distributive and commutative laws for dot product, and $|\vec{v}| = |\vec{u}|$.

Since the dot product is zero, the vectors are orthogonal, and $\angle C$ is a right angle, as required. □

Style notes:

- The proof is divided into paragraphs: the statement of the proposition; the setup translating the geometric hypotheses into vector notation; the main computation; and the conclusion.
- The proof does not depend on a diagram: the words make sense on their own, and one could draw a diagram based on them.
- No needed explanation is omited, and nothing else is inserted. For example, vector coordinates are irrelevant.
- For some reason, instead of the rays $\vec{CA}$, $\vec{CB}$ forming $\angle C$, every single student used the vectors $\vec{AC}, \vec{BC}$, which, when translated to start at C, form the vertical angle to $\angle C$, outside the triangle.