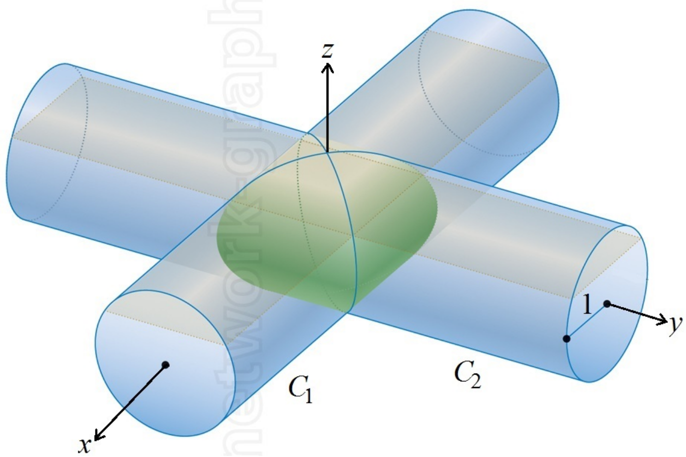
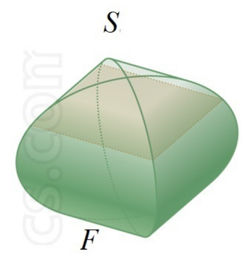
Math 254H **Models of Solids** **Mar 25, 2016**

We will construct paper models of some subtle solids enclosed by cylinder surfaces. Because a cylinder can be uncurled into a flat sheet of paper, the surfaces of our solids will correspond to certain shapes cut out of the sheet. These shapes will be defined by plane coordinates corresponding to arclengths on the curved surfaces.

1. Intersection of circular cylinders. Consider the solid circular cylinder C_1 consisting of the points within distance 1 from the x-axis; and similarly C_2 for the y-axis. Now define the solid $S = C_1 \cap C_2$, the region inside both cylinders.





a. PROBLEM: Give equations for the two cylinder surfaces, and describe S in the form:

$$\begin{gathered} a_1 \le z \le a_2 \\ b_1(z) \le y \le b_2(z) \\ c_1(y,z) \le x \le c_2(y,z). \end{gathered}$$

Use this to set up the triple integral and find the volume of S.

b. We want to determine the shape of the four identical surfaces of S, after they are uncurled (without stretching) onto a flat sheet. We focus on the front surface F which lies on C_2 and has $0 \le x \le 1$ and $-x \le y \le x$, with corners at the north and south poles of S. We can uncurl F onto a region R shaped like an eye centered at the origin, with left and right corners on the horizontal axis.

We draw two curves on F which will uncurl onto the axes of R:

- $\mathbf{p}(t) = (\cos(t), 0, \sin(t))$ for $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$, running vertically on F, uncurls to the horizontal axis (p-axis) of R.

- $\mathbf{q}(t) = (1, t, 0)$ for $-1 \le t \le 1$ uncurls to the vertical axis (q-axis) of R.

The curves $\mathbf{p}, \mathbf{q}$ intersect at $(1,0,0)$, the center of F, which corresponds to the origin $(p,q) = (0,0)$ in R. For a given point (x,y,z) on F, we determine its (p,q) coordinates as the distances traversed (arclengths) when moving on F from $(1,0,0)$ to (x,y,z), first along $\mathbf{p}$, then parallel to $\mathbf{q}$. (Since $\mathbf{q}$ is parallel to the y-axis, the q-coordinate in R is the same as the y-coordinate in F.)

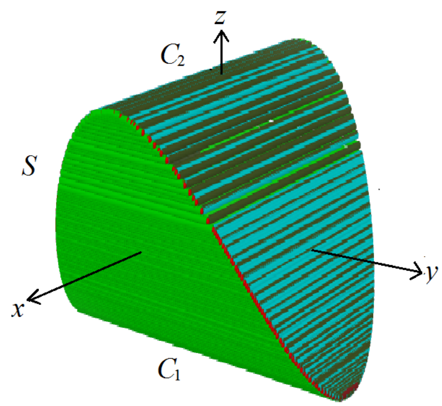
PROBLEM: Determine the (p,q) coordinates of a point along the right edge of F, reached by travelling upward from $\mathbf{p}(0)$ to $\mathbf{p}(t)$, then rightward parallel to $\mathbf{q}$ to touch the cylinder

surface C_1. Arclength formula: $L = \int |\mathbf{p}'(t)|\, dt$. The resulting p, q are functions of t, but write q as a function $q = c(p)$. Doing the same for the symmetric left edge point of F, you can describe the uncurled region R in the form $a \le p \le b$, $\ -c(p) \le q \le c(p)$.
Ans: $q = \pm c(p) = \pm \cos(p)$ for $-\frac{\pi}{2} \le p \le \frac{\pi}{2}$.

c. PROBLEM: To construct a paper model of solid S, print an outline of region R between the graphs $p = f(q)$ and $p = -f(q)$, making sure the vertical and horizontal axes are to scale, at least 10cm wide. (For example, plot both graphs with Wolfram|Alpha and print a screencap.) Trace the printout four times on card stock or thick paper, and cut out the four shapes. tape together the edges starting from the top, where four corners meet. Test your solid to see if the two horizontal profiles are perfect circles, and the vertical shadow is a square. Your solid should roll smoothly in both the x and y directions!

2. We repeat the above exercise with a new solid $S = C_1 \cap C_2$, the intersection of two *parabolic* cylinders defined by:

$$C_1 : \ z \ge x^2 - 2, \qquad\qquad C_2 : \ z \le 2 - y^2.$$



a. Set up and evaluate a triple integral in rectangular coordinates to find the volume of S. You will need some difficult antiderivatives: look these up on tables or W|A.

a′. Again find the volume using a triple integral in cylindrical coordinates $dz\, dr\, d\theta$. These coordinates are well adapted to the solid's circular xy-shadow.

b. Now again let F be the belly surface of S, which uncurls into a plane region R, and define axis-curves along F: $\mathbf{p}(t) = (t, 0, t^2 - 2)$ going down-and-up under the x-axis, and $\mathbf{q}(t) = (0, t, -2)$ going horizontally under the y-axis. For a fixed t, determine the arclength p from $\mathbf{p}(0)$ to $\mathbf{p}(t)$, and the distance q parallel to $\mathbf{q}(t)$ from $\mathbf{p}(t)$ to the right edge of F. You will need $\int \sqrt{1+u^2}\, du = \frac{1}{2} u\sqrt{1+u^2} + \frac{1}{2}\log\left(u + \sqrt{1+u^2}\right)$. This produces the coordinates (p, q) of the boundary of region R. These (p, q) depend on t, but you can eliminate t to write $p = c(q)$. (This happens to be simpler than the inverse function $q = c^{-1}(p)$.) Thus you get R in the form $a \le q \le b$, $\ -c(q) \le p \le c(q)$.
Ans: $p = \pm c(q) = \pm\left(\frac{1}{2}\sqrt{4-q^2}\sqrt{17-4q^2} + \frac{1}{2}\log(2\sqrt{4-q^2} + \sqrt{17-4q^2})\right)$, $-2 \le q \le 2$.

c. Print, trace, and cut out two copies of R, tape them together. This time the vertical shadow should be a circle, and the horizontal shadows parabolas.

NOTE: For the computations in 1(a), 1(b), you should not need a computer, and should write down all the computations by hand. For 2(a), 2(b), you should clearly set up the relevant integrals, but you may use the computer to integrate.