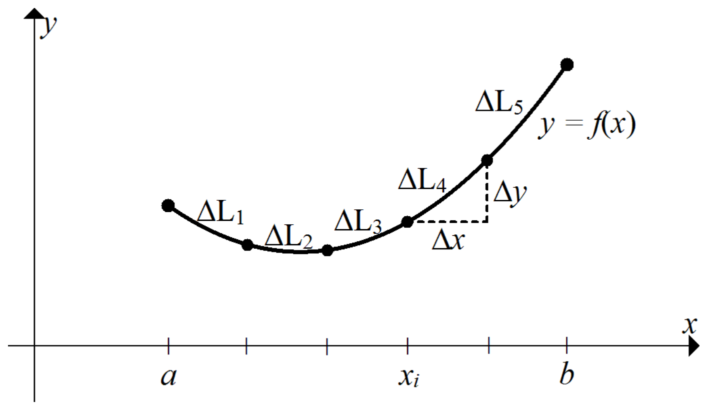
Math 133 # Arclength **Stewart §8.1**

Increments of length. In this section, we give an integral formula to compute the length of a curve, by the same Method of Slice Analysis we used in §5.2 to compute volume, and in §5.3 to compute work (see end §5.2).

We want the arclength L of a graph curve $y = f(x)$ for $x \in [a, b]$. We cut the curve into n bits determined by Δx-increments of $x \in [a, b]$. (In the picture, $n = 5$.)



Because the bit at the sample point x_i is so short, it is well approximated by a straight segment, and we can use the Pythagorean Theorem to compute its length:

$$\Delta L_i \;\approx\; \sqrt{(\Delta x)^2 + (\Delta y)^2}\,.$$

We want to write this as a term in a Riemann sum, so we must write it in the form $g(x_i)\,\Delta x$ for some function $g(x)$. We simply factor out Δx:

$$\Delta L_i \;\approx\; \sqrt{\left(1 + \tfrac{(\Delta y)^2}{(\Delta x)^2}\right)(\Delta x)^2} \;=\; \sqrt{1 + (\tfrac{\Delta y}{\Delta x})^2}\;\Delta x\,.$$

In the limit as $n \to \infty$, we get $\Delta x \to 0$ and $\frac{\Delta y}{\Delta x} \to \frac{dy}{dx} = f'(x_i)$, and the Riemann sum total of the ΔL_i's becomes an integral:

$$L \;=\; \lim_{n\to\infty} \sum_{i=1}^{n} \Delta L_i \;=\; \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + (\tfrac{\Delta y}{\Delta x})^2}\;\Delta x \;=\; \int_a^b \sqrt{1 + (\tfrac{dy}{dx})^2}\;dx\,.$$

In Newton notation:

$$L \;=\; \int_a^b \sqrt{1 + y'(x)^2}\;dx\,.$$

EXAMPLE: Compute the arclength of the curve $y = x\sqrt{x}$ over the interval $x \in [0, 4]$. We have $\frac{dy}{dx} = (x^{3/2})' = \frac{3}{2}x^{1/2}$, so:

$$L \;=\; \int_0^4 \sqrt{1 + (\tfrac{dy}{dx})^2}\;dx \;=\; \int_0^4 \sqrt{1 + \tfrac{9}{4}x}\;dx \;=\; \tfrac{8}{27}(1+\tfrac{9}{4}x)^{3/2}\Big|_{x=0}^{x=4} \;=\; \tfrac{8}{27}(\sqrt{10}-1) \;\approx\; 9.07$$

To check this, we compare with the straight-line distance between the endpoints $(0, 0)$ and $(4, 8)$: this is $\sqrt{4^2 + 8^2} \approx 8.9$, and indeed the length of the curve is slightly larger.

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EXAMPLE: Compute the circumference of the unit circle, which is twice the arclength of the graph $y = \sqrt{1-x^2}$ for $x \in [-1,1]$:

$$C \;=\; 2L \;=\; 2\int_{-1}^{1}\sqrt{1+(\tfrac{d}{dx}\sqrt{1-x^2}\,)^2}\;dx \;=\; 2\int_{-1}^{1}\sqrt{1+\Big(\tfrac{-2x}{2\sqrt{1-x^2}}\Big)^2}\;dx$$

$$=\;2\int_{-1}^{1}\sqrt{\frac{1-x^2+x^2}{1-x^2}}\;dx \;=\; 2\int_{-1}^{1}\frac{1}{\sqrt{1-x^2}}\;dx \;=\; 2\arcsin(x)\Big|_{x=-1}^{x=1} \;=\; 2\pi\,.$$

EXAMPLE: Compute the arclength of the parabola $y = x^2$ over any interval $x \in [0,b]$.

$$L \;=\; \int_0^b\sqrt{1+(\tfrac{d}{dx}(x^2))^2}\;dx \;=\; \int_0^b\sqrt{1+4x^2}\;dx\,.$$

To find the indefinite integral, we use the reverse trig substitution (§7.3): $x = \frac{1}{2}\tan(\theta)$, $\sqrt{1+4x^2} = \sec(\theta)$, $dx = \frac{1}{2}\sec^2(\theta)\,dx$:

$$\int\sqrt{1+4x^2}\;dx \;=\; \int\tfrac{1}{2}\sec^3(\theta)\,d\theta \;=\; \tfrac{1}{4}\ln\big|\tan(\theta)+\sec(\theta)\big| \;+\; \tfrac{1}{4}\tan(\theta)\sec(\theta),$$

where we use $\int\sec^3(\theta)\,d\theta$ from §7.2. Restoring the original variable, $\tan(\theta) = 2x$, $\sec(\theta) = \sqrt{1+4x^2}$, and taking the definite integral:

$$L \;=\; \Big[\tfrac{1}{4}\ln\Big|2x+\sqrt{1+4x^2}\,\Big| \;+\; \tfrac{1}{2}x\sqrt{1+4x^2}\,\Big]_{x=0}^{x=b} \;=\; \tfrac{1}{4}\ln\Big|2b+\sqrt{1+4b^2}\,\Big| \;+\; \tfrac{1}{2}b\sqrt{1+4b^2}\,.$$

Arclength tends to get quite complicated even for quite simple curves!

EXAMPLE: Compute the arclength of the curve $y = x^3$ over $x \in [0,1]$.

$$L \;=\; \int_0^1\sqrt{1+(\tfrac{d}{dx}(x^3))^2}\;dx \;=\; \int_0^1\sqrt{1+9x^4}\;dx\,.$$

This is already complicated enough that it has no algebraic antiderivative.*

Does this mean the arclength formula is useless? Not at all! We cannot get an answer on the algebraic level, but we can still get a numerical answer as accurate as we like. This means going from the integral formula for L back to the Riemann sums from which we deduced the integral. For example, taking $n = 1000$, the increment is $\Delta x = \frac{1}{1000} = 0.001$, and the sample points are $x_i = i\,\Delta x = (0.001)i$. The computer gives:

$$L \;\approx\; \sum_{i=1}^{n}\sqrt{1+9x_i^4}\;\Delta x \;=\; \sum_{i=1}^{1000}\sqrt{1+9(10^{-12})i^4}\;(0.001) \;\approx\; 1.548\,,$$

To gauge the accuracy of this, we re-do it with $n = 10{,}000$, getting $L \approx 1.547$, so we can be confident that $L \approx 1.54$ is accurate to 2 decimal places.

*The integral can be expressed in terms of an "elliptic function", but this is circular reasoning since elliptic functions themselves are defined as integrals!