

Reducing to standard trig forms. To find an indefinite integral $\int f(x) dx$, we transform it by methods like Substitution and Integration by Parts until we reduce to an integral we recognize from before, a “standard form”. In the previous section §7.2, we were able to compute most integrals involving products of trig functions, so these are now standard forms to work toward.

A common type of difficult integral involves forms like $\sqrt{\pm x^2 \pm 1}$. We convert such forms into trigonometric integrals, which at first seems to complicate them. However, we take careful advantage of the Pythagorean identities $\cos^2(\theta) + \sin^2(\theta) = 1$ and $\tan^2(\theta) + 1 = \sec^2(\theta)$, so that the resulting trig formulas simplify to previously known integrals.

Our first example is $\int \sqrt{1-x^2} dx$, which computes area under a unit semi-circle. This seems simple enough, but no obvious substitution or integration by parts will simplify it. The expression $\sqrt{1-x^2}$ reminds us of the identity $\sqrt{1-\sin^2(\theta)} = \cos(\theta)$, so imagine our integral was obtained from a more complicated one after a trig substitution $x = \sin(\theta)$: the current variable x is actually a function of a previous variable θ , and $dx = \cos(\theta) d\theta$. The previous integral would be:

$$\int \sqrt{1-x^2} dx = \int \sqrt{1-\sin^2(\theta)} \cdot \cos(\theta) d\theta, \quad \text{where } \begin{cases} x = \sin(\theta) \\ dx = \cos(\theta) d\theta. \end{cases}$$

Now this simplifies to a standard form from §7.2:*

$$\int \sqrt{1-\sin^2(\theta)} \cdot \cos(\theta) d\theta = \int \cos^2(\theta) d\theta = \frac{1}{2}\theta + \frac{1}{4}\sin(2\theta) = \frac{1}{2}\theta + \frac{1}{2}\sin(\theta)\cos(\theta).$$

Substituting back the original variable: $\theta = \arcsin(x)$, $\sin(\theta) = x$, $\cos(\theta) = \sqrt{1-x^2}$:

$$\int \sqrt{1-x^2} dx = \frac{1}{2} \arcsin(x) + \frac{1}{2} x \sqrt{1-x^2}.$$

Let’s check the area of the unit circle, i.e. twice the total area under $y = \sqrt{1-x^2}$:

$$2 \int_{-1}^1 \sqrt{1-x^2} dx = \left[\arcsin(x) + x \sqrt{1-x^2} \right]_{x=-1}^{x=1} = \arcsin(1) - \arcsin(-1) = \pi.$$

Integrals with $\sqrt{\pm x^2 \pm a^2}$. We choose a reverse trig substitution depending on the signs in the expression, then use the corresponding Pythagorean identity to obtain a standard trig form. See also the table of integrals further below.

$\sqrt{a^2-x^2}$	$x = a \sin(\theta)$	$dx = a \cos(\theta) d\theta$	$\sqrt{a^2-a^2 \sin^2(\theta)} = a \cos(\theta)$
$\sqrt{a^2+x^2}$	$x = a \tan(\theta)$	$dx = a \sec^2(\theta) d\theta$	$\sqrt{a^2+a^2 \tan^2(\theta)} = a \sec(\theta)$
$\sqrt{x^2-a^2}$	$x = a \sec(\theta)$	$dx = a \tan(\theta) \sec(\theta) d\theta$	$\sqrt{a^2 \sec^2(\theta)-a^2} = a \tan(\theta)$

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*Variable θ is irrelevant: $\int \cos^2(\theta) d\theta = \frac{1}{2}\theta + \frac{1}{4}\sin(2\theta)$ means the same as $\int \cos^2(x) dx = \frac{1}{2}x + \frac{1}{4}\sin(2x)$.

EXAMPLE: $\int \frac{1}{\sqrt{4-x^2}} dx$. Let $x = 2 \sin(\theta)$, $dx = 2 \cos(\theta) d\theta$, $\sqrt{4-(2 \sin(\theta))^2} = 2 \cos(\theta)$:

$$\int \frac{1}{\sqrt{4-x^2}} dx = \int \frac{1}{2 \cos(\theta)} \cdot 2 \cos(\theta) d\theta = \theta = \sin^{-1}\left(\frac{1}{2}x\right).$$

We could do this more directly by manipulating the integrand to the known derivative of $\arcsin(x)$ by the substitution $u = \frac{1}{2}x$:[†]

$$\int \frac{1}{\sqrt{4-x^2}} dx = \int \frac{1}{\sqrt{1-(\frac{1}{2}x)^2}} \cdot \frac{1}{2} dx = \int \frac{1}{\sqrt{1-u^2}} du = \sin^{-1}(u) = \sin^{-1}\left(\frac{1}{2}x\right).$$

EXAMPLE: $\int \frac{1}{\sqrt{9x^2+4}} dx = \int \frac{1}{\sqrt{(3x)^2+2^2}} dx$. Let $3x = 2 \tan(\theta)$, $dx = \frac{2}{3} \sec^2(\theta) d\theta$,

$\sqrt{9x^2+4} = \sqrt{(3x)^2+2^2} = \sqrt{(2 \tan(\theta))^2+2^2} = 2\sqrt{\tan^2(\theta)+1} = 2 \sec(\theta)$:

$$\begin{aligned} \int \frac{1}{\sqrt{9x^2+4}} dx &= \int \frac{1}{2 \sec(\theta)} \cdot \frac{2}{3} \sec^2(\theta) d\theta = \frac{1}{3} \int \sec(\theta) d\theta \\ &= \frac{1}{3} \ln|\tan(\theta) + \sec(\theta)| = \frac{1}{3} \ln\left|\frac{3}{2}x + \frac{1}{2}\sqrt{9x^2+4}\right|. \end{aligned}$$

Alternatively, we could use hyperbolic functions (§6.7), satisfying $\cosh^2(t) = \sinh^2(t) + 1$. Thus $x = \frac{2}{3} \sinh(t)$, $dx = \frac{2}{3} \cosh(t) dt$ and $\sqrt{9x^2+4} = 2\sqrt{\sinh^2(t)+1} = 2 \cosh(t)$ gives:

$$\begin{aligned} \int \frac{1}{\sqrt{9x^2+4}} dx &= \int \frac{1}{2 \cosh(t)} \frac{2}{3} \cosh(t) dt = \frac{1}{3} t \\ &= \frac{1}{3} \sinh^{-1}\left(\frac{3}{2}x\right) = \frac{1}{3} \ln\left(\frac{3}{2}x + \sqrt{\frac{9}{4}x^2+1}\right). \end{aligned}$$

A third method is to substitute $u = \frac{3}{2}x$ to get $\frac{1}{3} \int \frac{1}{\sqrt{u^2+1}} du = \frac{1}{3} \sinh^{-1}(u) = \frac{1}{3} \sinh^{-1}\left(\frac{3}{2}x\right)$.

EXAMPLE: $\int \frac{\sqrt{x^2-25}}{x} dx$; $x = 5 \sec(\theta)$, $dx = 5 \tan(\theta) \sec(\theta) d\theta$, $\sqrt{(5 \sec(\theta))^2-25} = 5 \tan(\theta)$:

$$\begin{aligned} \int \frac{\sqrt{x^2-25}}{x} dx &= \int \frac{5 \tan(\theta)}{5 \sec(\theta)} \cdot 5 \tan(\theta) \sec(\theta) d\theta = 5 \int \tan^2(\theta) d\theta \\ &= 5 \int \sec^2(\theta) - 1 d\theta = 5 \tan(\theta) - 5\theta = \sqrt{x^2-25} - 5 \sec^{-1}\left(\frac{1}{5}x\right). \end{aligned}$$

Rewriting $\sec^{-1}(u) = \tan^{-1}\sqrt{u^2-1}$ as in §6.6, this becomes: $\sqrt{x^2-25} - 5 \tan^{-1}\left(\frac{1}{5}\sqrt{x^2-25}\right)$.

EXAMPLE: $\int \frac{1}{(x^2-4)^{3/2}} dx$; $x = 2 \sec(\theta)$, $dx = 2 \tan(\theta) \sec(\theta) d\theta$, $((2 \sec(\theta))^2-4)^{3/2} = 8 \tan^3(\theta)$:

$$\begin{aligned} \int \frac{1}{(x^2-4)^{3/2}} dx &= \int \frac{1}{8 \tan^3(\theta)} \cdot 2 \tan(\theta) \sec(\theta) d\theta = \frac{1}{4} \int \sin^{-2}(\theta) \cos(\theta) d\theta \\ &= -\frac{1}{4} \sin(\theta)^{-1} = -\frac{1}{4} \frac{\frac{1}{2}x}{\sqrt{(\frac{1}{2}x)^2-1}} = -\frac{x}{4\sqrt{x^2-4}}. \end{aligned}$$

Or: $x = 2 \cosh(t)$, $x^2-4 = 4 \sinh^2(t)$ produces $\frac{1}{4} \int \operatorname{csch}^2(t) dt = -\frac{1}{4} \coth(t) = -\frac{\cosh(t)}{4 \sinh(t)}$.

[†]See §6.6 for inverse trig functions like $\arcsin = \sin^{-1}$, and their derivatives.

EXAMPLE: $\int \sqrt{x^2 - 1} dx$; $x = \cosh(t)$, $dx = \sinh(t) dt$, $x^2 - 1 = \cosh^2(t) - 1 = \sinh^2(t)$:

$$\begin{aligned} \int \sqrt{x^2 - 1} dx &= \int \sqrt{\cosh^2(t) - 1} \sinh(t) dt \\ &= \int \sinh^2(t) dt = \frac{1}{2} \cosh(t) \sinh(t) - \frac{1}{2} t \\ &= \frac{1}{2} x \sqrt{x^2 - 1} - \frac{1}{2} \cosh^{-1}(x), \\ &= \frac{1}{2} x \sqrt{x^2 - 1} - \frac{1}{2} \ln(x + \sqrt{x^2 - 1}), \end{aligned}$$

using the formulas for $\int \sinh^2(t) dt$ and $\cosh^{-1}(x)$ from §6.7.

EXAMPLE: $\int \sqrt{1 - x^2} dx$; $x = \tanh(t)$, $dx = \operatorname{sech}^2(t) dt$, $1 - x^2 = 1 - \tanh^2(t) = \operatorname{sech}^2(t)$:

$$\begin{aligned} \int \sqrt{1 - x^2} dx &= \int \operatorname{sech}^3(t) dt = \frac{1}{2} \left(\tanh(t) \operatorname{sech}(t) + \int \operatorname{sech}(t) dt \right) \\ &= \frac{1}{2} \tanh(t) \operatorname{sech}(t) + \tan^{-1}(e^t) = \frac{1}{2} x \sqrt{1 - x^2} + \tan^{-1} \sqrt{\frac{1+x}{1-x}}, \end{aligned}$$

via §6.7, 7.2. This simplifies to the previous $\int \sqrt{1 - x^2} dx = \frac{1}{2} x \sqrt{1 - x^2} + \frac{1}{2} \sin^{-1}(x) + \frac{\pi}{4}$.

EXAMPLE: $\int \frac{1}{x\sqrt{x^2+x}} dx$. It will not help to complete the square and do a linear substitution $u = ax + b$, since the denominator will still contain the factor $x = \frac{1}{a}u - b$. Rather, write $x^2 + x = x^2(1 + \frac{1}{x})$ and do a linear fractional substitution $u = 1 + \frac{1}{x}$, $du = -\frac{1}{x^2} dx$:

$$\int \frac{1}{x\sqrt{x^2+x}} dx = \int \frac{1}{x^2 \sqrt{1 + \frac{1}{x}}} dx = - \int \frac{1}{\sqrt{u}} du = -2\sqrt{u} = -2\sqrt{1 + \frac{1}{x}}.$$

EXAMPLE: $\int \frac{1}{x\sqrt{3x^2-2x-1}} dx$. Using the strategy of the previous example, we rewrite as a quadratic in $\frac{1}{x}$ and complete the square in terms of $u = \frac{a}{x} + b$:

$$\begin{aligned} 3x^2 - 2x - 1 &= x^2 \left(3 - \frac{2}{x} - \frac{1}{x^2} \right) \\ &= x^2 \left(3 + 1^2 - 1^2 - 2(1)\frac{1}{x} - \frac{1}{x^2} \right) \\ &= x^2 \left(4 - \left(1 + \frac{1}{x} \right)^2 \right) = 4x^2 \left(1 - \left(\frac{1}{2} + \frac{1}{2x} \right)^2 \right). \end{aligned}$$

Then $u = \frac{1}{2x} + \frac{1}{2}$, $du = -\frac{1}{2x^2} dx$ gives:

$$\int \frac{1}{x\sqrt{3x^2-2x-1}} dx = \int \frac{1}{2x^2 \sqrt{1 - (\frac{1}{2} + \frac{1}{2x})^2}} dx = \int \frac{-1}{\sqrt{1-u^2}} du = -\sin^{-1}(u) = -\sin^{-1}\left(\frac{1}{2x} + \frac{1}{2}\right).$$

EXAMPLE: $\int \sqrt{\frac{x}{x-1}} dx$. It turns out best to substitute $u = \sqrt{x-1}$, $x = u^2 + 1$, $dx = 2u du$:

$$\begin{aligned} \int \sqrt{\frac{x}{x-1}} dx &= \int \frac{\sqrt{u^2+1}}{u} 2u du = \int 2\sqrt{u^2+1} du = u\sqrt{u^2+1} + \sinh^{-1}(u) \\ &= \sqrt{x(x-1)} + \sinh^{-1}\sqrt{x-1} = \sqrt{x(x-1)} + \ln(\sqrt{x} + \sqrt{x-1}) \end{aligned}$$

EXAMPLE: $\int \frac{1}{\sqrt{x^2+x}} dx$. In §6.7, completing the square gave $\cosh^{-1}(2x+1) = \ln(2x+1+2\sqrt{x^2+x})$, but sneaky substitution gives a surprise equivalent answer: $z = \sqrt{x}$, $x = z^2$, $dx = 2z dz$;

$$\int \frac{1}{\sqrt{x^2+x}} dx = \int \frac{1}{\sqrt{z^4+z^2}} 2z dz = \int \frac{2}{\sqrt{z^2+1}} dz = 2 \sinh^{-1}(z) = 2 \sinh^{-1}\sqrt{x}.$$

EXAMPLE: $\int \sec^3(x) dx$. We did this notorious example in §7.2 via integration by parts, but double trig-hyperbolic substitution also works. We have $\sec(x) = \sqrt{1+\tan^2(x)}$, which is similar to the hyperbolic identity $\cosh(t) = \sqrt{1+\sinh^2(t)}$. Thus, we substitute:

$$\begin{aligned} \tan(x) &= \sinh(t), & \sec^2(x) dx &= \cosh(t) dt, \\ \int \sec^3(x) dx &= \int \sqrt{1+\tan^2(x)} \sec^2(x) dx = \int \sqrt{1+\sinh^2(t)} \cosh(t) dt \\ &= \int \cosh^2(t) dt = \int \left(\frac{1}{2} \cosh(2t) + \frac{1}{2}\right) dt = \frac{1}{4} \sinh(2t) + \frac{1}{2} t \\ &= \frac{1}{2} \sinh(t) \cosh(t) + \frac{1}{2} t = \frac{1}{2} \tan(x) \sec(x) + \frac{1}{2} \sinh^{-1}(\tan(x)). \end{aligned}$$

Then $\sinh^{-1}(x) = \ln(x + \sqrt{1+x^2})$ recovers the previous answer:

$$\int \sec^3(x) dx = \frac{1}{2} \tan(x) \sec(x) + \frac{1}{2} \ln(\tan(x) + \sec(x)).$$

Table of integrals which produce inverse trig and hyperbolic functions (omitting $+C$).

$$\begin{aligned} \int \frac{1}{\sqrt{1-x^2}} dx &= \sin^{-1}(x) = \frac{\pi}{2} - \cos^{-1}(x) & \int \frac{1}{\sqrt{x^2-1}} dx &= \cosh^{-1}(x) = \ln(x + \sqrt{x^2-1}) \\ \int \frac{1}{\sqrt{1+x^2}} dx &= \sinh^{-1}(x) = \ln(x + \sqrt{1+x^2}) \\ \int \frac{1}{x\sqrt{x^2-1}} dx &= \sec^{-1}(x) = \tan^{-1}\sqrt{x^2-1}^\dagger & \int \frac{1}{x\sqrt{1-x^2}} dx &= -\operatorname{sech}^{-1}(x) = \ln(x) - \ln(1 + \sqrt{1-x^2}) \\ \int \frac{1}{x\sqrt{1+x^2}} dx &= -\operatorname{csch}^{-1}(x) = \ln(x) - \ln(1 + \sqrt{1+x^2}) \\ \int \frac{1}{1+x^2} dx &= \tan^{-1}(x) & \int \frac{1}{1-x^2} dx &= \tanh^{-1}(x) = \frac{1}{2} \ln(1+x) - \frac{1}{2} \ln(1-x) \\ \int \sqrt{1-x^2} dx &= \frac{1}{2} x \sqrt{1-x^2} + \frac{1}{2} \sin^{-1}(x) & \int \sqrt{x^2-1} dx &= \frac{1}{2} x \sqrt{x^2-1} - \frac{1}{2} \cosh^{-1}(x) \\ \int \sqrt{x^2+1} dx &= \frac{1}{2} x \sqrt{x^2+1} + \frac{1}{2} \sinh^{-1}(x) \end{aligned}$$

[†]Equality holds for $x \geq 1$; for $x \leq -1$, see end of §6.6.