

Review of integrals. The definite integral gives the cumulative total of many small parts, such as the slivers which add up to the area under a graph. Numerically, it is a limit of Riemann sums:

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x,$$

where we divide the interval $x \in [a, b]$ into n increments of size $\Delta x = \frac{b-a}{n}$ with division points $a < a+\Delta x < a+2\Delta x < \dots < a+n\Delta x = b$, and x_1, \dots, x_n are sample points from each increment. This definition is not a theoretical curiosity: it is the reason integrals are relevant to physical problems, and it is the only way to evaluate most integrals: there is no algebraic way.

However, for sufficiently simple functions $f(x)$, we can evaluate integrals algebraically by the shortcut of the Second Fundamental Theorem of Calculus. This says that if $f(x)$ is the rate of change of some known antiderivative $F(x)$, then the integral of $f(x)$ is the cumulative total change of $F(x)$:

$$F'(x) = f(x) \quad \Longrightarrow \quad \int_a^b f(x) dx = F(x)|_{x=a}^{x=b} = F(b) - F(a).$$

(The First Fundamental Theorem says that the definite integral gives an antiderivative even if there is no formula $F(x)$: defining $I(x) = \int_a^x f(t) dt$, we have $I'(x) = f(x)$.)

Algebraic integration is the process of finding antiderivative formulas, denoted as indefinite integrals $\int f(x) dx = F(x) + C$. The most direct method is to reverse Basic Derivatives, such as $(x^p)' = px^{p-1}$ reversing to $\int x^p dx = \frac{x^{p+1}}{p+1}$. Our only other integration method so far is the Substitution Method, which reverses the Chain Rule:

$$\int f(g(x)) g'(x) dx = \int f(u) du = F(u) + C = F(g(x)) + C,$$

where $u = g(x)$, $du = g'(x) dx$, and $F'(u) = f(u)$.

Reversing the Product Rule. Since we have:

$$(f(x)g(x))' = f(x)g'(x) + g(x)f'(x),$$

we can take the antiderivative of both sides to give:

$$\begin{aligned} f(x)g(x) &= \int f(x)g'(x) dx + \int g(x)f'(x) dx, \\ \int f(x)g'(x) dx &= f(x)g(x) - \int g(x)f'(x) dx. \end{aligned}$$

In Leibnitz notation, taking $u = f(x)$, $du = f'(x) dx$ and $v = g(x)$, $dv = g'(x) dx$:

$$\int u dv = uv - \int v du.$$

This method transforms the integral of a product $f(x)g'(x)$ into $f(x)g(x)$ minus the integral of $g(x)f'(x)$, the other term in the Product Rule; we can think of lowering $f(x)$ to its derivative $f'(x)$ and raising $g'(x)$ to its antiderivative $g(x)$.

Method for Integration by Parts.

1. Given an indefinite integral $\int h(x) dx$, find a factor of the integrand $h(x)$ which you recognize as the derivative of a function $g(x)$: that is, write $h(x) = f(x) \cdot g'(x)$.
2. Taking $u = f(x)$, $dv = g'(x) dx$, transform the integral $\int u dv = \int f(x) g'(x) dx$ into $uv - \int v du = f(x)g(x) - \int g(x) f'(x) dx$.
3. Simplify $g(x) f'(x)$, possibly using identities, and try to find its integral by other methods such as Substitution.
4. Sometimes you can repeat Steps 1 & 2 on $\int g(x) f'(x) dx$ with a *different* u, v .^{*} This might further simplify integral so you can evaluate by other methods.
5. Instead of simplifying the integral, Step 3 or 4 might give an expression with the *same* integral you started with. Solve the resulting equation to find that integral.

Notice that Step 1 is the same as for the Method of Substitution, where you want to find a factor of the integrand which is a known derivative $g'(x)$; but for Substitution, $g(x)$ should also appear as an inside function in the remaining factor: $h(x) = f(g(x)) \cdot g'(x)$.

EXAMPLE: Evaluate $\int x \cos(x) dx$. There are two obvious candidates for u, v . First, if we take $u = \cos(x)$, $dv = x dx$, we get $du = -\sin(x) dx$, $v = \frac{1}{2}x^2$, and:

$$\begin{aligned} \int u dv &= uv - \int v du \\ \int \cos(x) x dx &= \cos(x) \left(\frac{1}{2}x^2\right) - \int \frac{1}{2}x^2 (-\sin(x)) dx \end{aligned}$$

Unfortunately, the new integral $\int x^2 \sin(x) dx$ is harder than the original $\int x \cos(x) dx$. We must make a wiser choice of u, v , so that the derivative du will be simpler than the original u , while the antiderivative v will be no worse than the original dv .

The other obvious choice will work: take $u = x$, $dv = \cos(x) dx$, so that $du = 1 dx$ and $v = \sin(x)$. Then:[†]

$$\begin{aligned} \int u dv &= uv - \int v du \\ \int x \cos(x) dx &= x \sin(x) - \int \sin(x) 1 dx \\ &= x \sin(x) + \cos(x). \end{aligned}$$

Thus, Steps 1–3 were enough to integrate.

To check our answer, we reverse our Integration by Parts using the Product Rule:

$$\begin{aligned} (x \sin(x) + \cos(x))' &= x \sin'(x) + \sin(x)(x)' + \cos'(x) \\ &= x \cos(x) + \sin(x) - \sin(x) = x \cos(x). \end{aligned}$$

EXAMPLE: Evaluate $\int x^2 e^{-x} dx$. We should choose $u = x^2$, $dv = e^{-x}$, so that $du = 2x dx$ is simpler, but $v = -e^{-x}$ is no more complicated:

$$\begin{aligned} \int u dv &= uv - \int v du \\ \int x^2 e^{-x} dx &= x^2(-e^{-x}) - \int (-e^{-x}) 2x dx \\ &= -x^2 e^{-x} + 2 \int x e^{-x} dx \end{aligned}$$

^{*}Repeating with the *same* factorization $\int v du$ would get back the original integral $\int u dv$.

[†]For brevity, we neglect the constant $+C$, though you should write it in formal work.

Going on to Step 4, we repeat the process for the integral on the right side, this time with $u = x$, $dv = e^{-x} dx$ and $du = dx$, $v = -e^{-x}$:

$$\begin{aligned}\int u dv &= uv - \int v du \\ \int x e^{-x} dx &= x(-e^{-x}) - \int (-e^{-x}) dx \\ &= -xe^{-x} + \int e^{-x} dx \\ &= -xe^{-x} + (-e^{-x})\end{aligned}$$

Putting these together:

$$\int x^2 e^{-x} dx = -x^2 e^{-x} + 2(-xe^{-x} - e^{-x}) = -(x^2 + 2x + 2)e^{-x}.$$

EXAMPLE: Evaluate $\int e^x \sin(x) dx$. Steps 1–4 give:

$$\begin{aligned}\int e^x \sin(x) dx &= e^x \sin(x) - \int e^x \cos(x) dx, & u = \sin(x), v = e^x \\ &= e^x \sin(x) - (e^x \cos(x) - \int e^x (-\sin(x)) dx), & u = \cos(x), v = e^x.\end{aligned}$$

We conclude:

$$\int e^x \sin(x) dx = e^x \sin(x) - e^x \cos(x) - \int e^x \sin(x) dx.$$

Since our integral $\int e^x \sin(x) dx$ appears on both sides, we go to Step 5 and solve for it:

$$\int e^x \sin(x) dx = \frac{1}{2}(e^x \sin(x) - e^x \cos(x)).$$

EXAMPLE: Evaluate $\int \ln(x) dx$. Here there does not seem to be any dv factor, but we can always take $dv = 1 dx$, so $v = x$:

$$\begin{aligned}\int u dv &= uv - \int v du \\ \int \ln(x) 1 dx &= \ln(x) x - \int x \frac{1}{x} dx \\ &= x \ln(x) - x.\end{aligned}$$

EXAMPLE: Evaluate $\int \sin^{-1}(x) dx$.[‡] Again we must use $u = \sin^{-1}(x)$ and $dv = 1 dx$, counting on the fact that du is simpler than u :

$$\begin{aligned}\int u dv &= uv - \int v du \\ \int \sin^{-1}(x) 1 dx &= \sin^{-1}(x) x - \int x \frac{1}{\sqrt{1-x^2}} dx\end{aligned}$$

Continuing Step 3, we use the substitution $z = 1 - x^2$ on the right-hand integral:

$$\int x \frac{1}{\sqrt{1-x^2}} dx = -\frac{1}{2} \int \frac{1}{\sqrt{1-x^2}} (-2x) dx = -\frac{1}{2} \int \frac{1}{\sqrt{z}} dz = -\sqrt{z} = -\sqrt{1-x^2}.$$

Combining:

$$\int \sin^{-1}(x) dx = x \sin^{-1}(x) - (-\sqrt{1-x^2}) = x \sin^{-1}(x) + \sqrt{1-x^2}.$$

[‡]Notation: $\sin^{-1}(x) = \arcsin(x)$, but $\sin(x)^{-1} = \frac{1}{\sin(x)} = \csc(x)$.