Math 133

Method for Integration

§7.1–7.4

Given a function \( f(x) \), we wish to find the indefinite integral \( \int f(x) \, dx = F(x) + C \), i.e., an antiderivative function with \( F'(x) = f(x) \). For brevity, we omit the constant \( +C \).

1. Basic integrals which directly reverse basic derivatives:

\[
\int x^p \, dx = \frac{1}{p+1} x^{p+1} \quad (p \neq -1) \quad \int \frac{1}{x} \, dx = \ln|x| \quad \int e^x \, dx = e^x
\]

\[
\int \sin(x) \, dx = -\cos(x) \quad \int \cos(x) \, dx = \sin(x)
\]

\[
\int \sec^2(x) \, dx = \tan(x) \quad \int \tan(x) \sec(x) \, dx = \sec(x)
\]

\[
\int \frac{1}{\sqrt{1-x^2}} \, dx = \arcsin(x) \quad \int \frac{1}{1+x^2} \, dx = \arctan(x)
\]

2. Substitution: Factor the integrand so that \( \int f(x) \, dx = \int h(g(x)) \cdot g'(x) \, dx \).

Take \( u = g(x) \), \( du = g'(x) \, dx \), so that \( \int h(g(x)) \cdot g'(x) \\, dx = \int h(u) \, du \). Integrate to get \( \int f(x) \, dx = H(u) \). Restore the original variable: \( \int f(x) \, dx = H(g(x)) \).

Tips: Take an inside function \( u = g(x) \); if there is no factor \( du = g'(x) \, dx \), multiply by \( \frac{1}{g'(x)} \cdot g'(x) \), or take inverse function \( x = g^{-1}(u) \), \( dx = (g^{-1})'(u) \, du \). Or start with a factor \( du = g'(x) \, dx \) and in the other factor, reverse-substitute \( x = g^{-1}(u) \).

3. Integration by Parts. Find a known derivative \( g'(x) \) as a factor of the integrand:

\[ \int f(x) \, dx = \int h(x) \cdot g'(x) \, dx = h(x) \cdot g(x) - \int h(x) \cdot g''(x) \, dx, \text{ i.e. } \int u \, dv = uv - \int v \, du. \]

The remaining integral \( \int g'(x) \cdot h''(x) \, dx \) should be easier, provided \( h''(x) \) is simpler than \( h(x) \), but \( g'(x) \) is about as complicated as \( g'(x) \). Or try \( g'(x) = 1, \, g(x) = x \).

After identities, the right side may again contain \( -\int f(x) \, dx \): solve for the integral.

4. Products of Trig Functions. Substitute by factoring out a derivative \( g'(x) = \cos(x), \sin(x), \sec^2(x) \) or \( \tan(x) \cdot \sec(x) \); write remaining factor in terms of \( u = g(x) \) using \( \cos^2(x) + \sin^2(x) = 1 \), \( \tan^2(x) + 1 = \sec^2(x) \), \( \tan(x) = \frac{\sin(x)}{\cos(x)} \), \( \sec(x) = \frac{1}{\cos(x)} \).

Otherwise, use identities \( \sin^2(x) = \frac{1}{2} - \frac{1}{2} \cos(2x) \), \( \cos^2(x) = \frac{1}{2} + \frac{1}{2} \cos(2x) \).

A hard case: \( \int \sec(x) \, dx = \ln[\tan(x) + \sec(x)] \). Geometric substitution converts any trig integral to rational: \( \cos(\theta) = \frac{1-t^2}{1+t^2}, \sin(\theta) = \frac{2t}{1+t^2}, \, d\theta = \frac{2}{1+t^2} \, dt, \, t = \tan(\frac{1}{2}\theta) \).

5. Reverse Trig Substitution. If \( \sqrt{a^2-x^2} \) appears in \( \int f(x) \, dx \), complicate it by substituting \( x = a \sin(\theta), \, dx = a \cos(\theta) \, d\theta \); simplify \( \sqrt{a^2-x^2} = \sqrt{a^2-(a \sin(\theta))^2} = a \cos(\theta) \).

Do the resulting trig integral; then restore by \( \theta = \sin^{-1}(\frac{x}{a}) \).

For \( \sqrt{x^2-a^2} \), use \( x = a \sec(\theta) \) or \( a \cosh(t) \); for \( \sqrt{x^2+a^2} \), use \( x = a \tan(\theta) \) or \( a \sinh(t) \).

6. Partial Fractions integrates rational functions \( f(x) = \frac{g(x)}{h(x)} = \text{polynomial} \), \( \frac{r(x)}{h(x)} \).

If \( g(x) \) has degree greater than or equal to \( h(x) \), perform long division to get \( f(x) = q(x) + \frac{r(x)}{h(x)} \), where \( r(x) \) has degree less than \( h(x) \), and proceed with \( \frac{r(x)}{h(x)} \).

If the denominator factors as \( h(x) = (x-a)/(x-b) \cdots \) with \( a, b, \ldots \) all different, split \( f(x) \) into the form: \( f(x) = \frac{g(x)}{h(x)} = \frac{A}{x-a} + \frac{B}{x-b} + \cdots \). Solve for the constant \( A \) after clearing denominators and substituting \( x = a \); and similarly for the other constants \( B, \ldots \). Finally, integrate using \( \int \frac{A}{x-a} \, dx = A \ln|x-a| = \ln(|x-a|^A) \).

See §7.4 if \( h(x) \) has factors \( (x-a)^n \) or quadratic \( ax^2+bx+c = (x-k)^2 + \ell, a, \ell \geq 0 \).

7. An integral has no elementary formula if it reduces to one of these special functions: sine integral \( \text{Si}(x) = \int \frac{\sin(x)}{x} \, dx \); exponential integral \( \text{Ei}(x) = \int e^{-x} \, dx \); error function \( \text{erf}(x) = \int e^{-x^2} \, dx \); logarithmic integral \( \text{Li}(x) = \int \frac{1}{\ln(x)} \, dx \); dilogarithm \( \text{Li}_2(x) = -\int \frac{\ln(1-x)}{x} \, dx \); elliptic integrals \( \int \frac{1}{\sqrt{1-k^2 \sin^2(\theta)}} \, d\theta \) and \( \int \sqrt{1-k^2 \sin^2(\theta)} \, d\theta \) for \( k \neq 1 \).

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