L'Hôpital's Rule

This technique evaluates limits which approach indeterminate forms like $\frac{0}{0}$ and $\frac{\infty}{\infty}$.

> *Theorem:* For functions $f(x), g(x)$, suppose $f'(x), g'(x)$ exist and $g'(x) \neq 0$, on some interval $x \in (a{-}\delta, a{+}\delta)$. Suppose that either:
>
> $$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0 \quad \text{or} \quad \lim_{x\to a} |f(x)| = \lim_{x\to a} |g(x)| = \infty.$$
>
> Then:
> $$\lim_{x\to a} \frac{f(x)}{g(x)} \;=\; \lim_{x\to a} \frac{f'(x)}{g'(x)},$$
> provided the right side limit exists, or equals ∞ or $-\infty$.
>
> This also holds with $x \to a$ replaced by a one-sided limit $x \to a^+$ or $x \to a^-$.

There is another version for limits as x becomes very large:

> *Theorem:* Let $f(x), g(x)$ be functions which are differentiable and $g'(x) \neq 0$, on a semi-infinite interval $x \in (c, \infty)$. Suppose that either:
>
> $$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = 0 \quad \text{or} \quad \lim_{x\to\infty} |f(x)| = \lim_{x\to\infty} |g(x)| = \infty.$$
>
> Then:
> $$\lim_{x\to\infty} \frac{f(x)}{g(x)} \;=\; \lim_{x\to\infty} \frac{f'(x)}{g'(x)},$$
> provided the right side limit exists, or equals ∞ or $-\infty$.
>
> This also holds with $x \to \infty$ replaced with $x \to -\infty$.

*Proof.** There is an easy and enlightening proof of the Theorem if we assume:

$$\lim_{x\to a} f(x) = f(a) = 0, \qquad \lim_{x\to a} g(x) = g(a) = 0,$$

$$\lim_{x\to a} f'(x) = f'(a), \qquad \lim_{x\to a} g'(x) = g'(a) \neq 0.$$

In this case:

$$\lim_{x\to a} \frac{f'(x)}{g'(x)} \;=\; \frac{f'(a)}{g'(a)} \;=\; \lim_{x\to a} \frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}} \;=\; \lim_{x\to a} \frac{f(x)-f(a)}{g(x)-g(a)} \;=\; \lim_{x\to a} \frac{f(x)}{g(x)}.$$

That is, the quotient on the left is approximately $\frac{\Delta f}{\Delta g}$. But if f starts at $f(a) = 0$, then the change in $f(x)$ is just the value of $f(x)$: that is, $\Delta f = f(x) - f(a) = f(x)$; and similarly $\Delta g = g(x)$.

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*A more complete proof. Assume only that $\lim_{x\to a} f(x) = f(a) = 0$, $\lim_{x\to a} g(x) = g(a) = 0$ and $\lim_{x\to a} f'(x)/g'(x)$ exists, redefining $f(a), g(a)$ if necessary. This means $f'(x), g'(x)$ are defined and $g'(x) \neq 0$ near $x = a$. I claim that also $g(x) \neq 0$ near $x = a$. Otherwise, if we had $g(x) = 0$ arbitrarily near $x = a$, the Mean Value Theorem (§3.2) would imply $g'(c) = 0$ for $c \in (a, x)$ or (x, a), a contradiction.

The Cauchy Mean Value Theorem (§3.2 end) says if $f(x), g(x)$ are continuous on $[a, b]$ and differentiable on (a, b), then there is $c \in (a, b)$ with $\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$, provided denominators are non-zero. Apply this to a sufficiently small interval $[a, x]$ or $[x, a]$ getting $c_x \in (a, x)$ or (x, a) with $f(x)/g(x) = f'(c_x)/g'(c_x)$. Now, as $x \to a$, also $c_x \to a$, and $f(x)/g(x) = f'(c_x)/g'(c_x)$ approaches the same value as $f'(x)/g'(x)$.

EXAMPLE: $\lim_{x\to 2}\frac{x-2}{x^2-4}$. The top and bottom both approach zero, so the limit approaches the indeterminate form $\frac{0}{0}$, and L'Hôpital's Rule applies.

$$\lim_{x\to 2}\frac{x-2}{x^2-4} \ \overset{\text{Hôp}}{=}\ \lim_{x\to 2}\frac{(x-2)'}{(x^2-4)'} \ =\ \lim_{x\to 2}\frac{1}{2x} \ =\ \frac{1}{4}.$$

In this simple case, we can also find the limit by cancelling vanishing factors in the numerator and denominator:

$$\lim_{x\to 2}\frac{x-2}{x^2-4} \ =\ \lim_{x\to 2}\frac{x-2}{(x-2)(x+2)} \ =\ \lim_{x\to 2}\frac{1}{x+2} \ =\ \frac{1}{4}\,.$$

Similar reasoning would apply to the $\frac{\infty}{\infty}$ form $\lim\limits_{x\to\infty}\frac{x-2}{x^2-4} \overset{\text{Hôp}}{=} \lim\limits_{x\to\infty}\frac{1}{2x} = 0$.

EXAMPLE: $\lim_{x\to 0}\frac{e^x-1-x}{x^2}$. This approaches $\frac{0}{0}$, so L'Hôpital applies.

$$\lim_{x\to 0}\frac{e^x-1-x}{x^2} \ \overset{\text{Hôp}}{=}\ \lim_{x\to 0}\frac{e^x-0-1}{2x}\,.$$

This still approaches $\frac{0}{0}$, so we can use L'Hôpital again:

$$\lim_{x\to 0}\frac{e^x-0-1}{2x} \ \overset{\text{Hôp}}{=}\ \lim_{x\to 0}\frac{e^x}{2} \ =\ \frac{e^0}{2} \ =\ \frac{1}{2}\,.$$

EXAMPLE: $\lim_{x\to 0^+} x\ln(x)$. (Here we use a one-sided limit $x\to 0^+$ because $\ln(x)$ is undefined for $x<0$.) This approaches the indeterminate form $0\cdot(-\infty)$, so it is a difficult limit, but we must manipulate it into a quotient to apply L'Hôpital:

$$\lim_{x\to 0^+} x\ln(x) \ =\ \lim_{x\to 0^+}\frac{\ln(x)}{1/x}$$

Now top and bottom become infinite approaching $\frac{-\infty}{\infty}$, so L'Hôpital applies.

$$\lim_{x\to 0^+} x\ln(x) \ =\ \lim_{x\to 0^+}\frac{\ln(x)}{1/x} \ \overset{\text{Hôp}}{=}\ \lim_{x\to 0^+}\frac{1/x}{-1/x^2} \ =\ \lim_{x\to 0^+}(-x) \ =\ 0\,.$$

EXAMPLE: $\lim_{x\to 0} x^x$. This approaches the indeterminate form 0^0, but we can once again manipulate it into a limit we can handle:

$$\lim_{x\to 0} x^x \ =\ \lim_{x\to 0} e^{\ln(x)\,x} \ =\ \lim_{x\to 0}\exp(x\ln(x)) \ =\ \exp\Big(\lim_{x\to 0} x\ln(x)\Big)\,.$$

We can move the limit inside exp( ) because it is a continuous function (see §1.8 Composition Law). Applying the previous example, the limit becomes $\exp(0)=1$.

EXAMPLE: $\lim_{x\to 0}\frac{\sin(x)}{e^x}$. The bottom does not approach 0, so this is not indeterminate at all, and *L'Hôpital does not apply here.* Instead, this is an easy limit that can be evaluated by continuity (plugging in):

$$\lim_{x\to 0}\frac{\sin(x)}{e^x} \ =\ \frac{\sin(0)}{e^0} \ =\ \frac{0}{1} \ =\ 0.$$

If we incorrectly try to apply L'Hôpital when it is not valid, we get a wrong answer:

$$\lim_{x\to 0}\frac{\sin(x)}{e^x}\ ??\overset{\text{Hôp}}{=}??\ \lim_{x\to 0}\frac{\cos(x)}{e^x} \ =\ \frac{\cos(0)}{e^0} \ =\ 1 \text{ (WRONG)}.$$

EXAMPLE: $\lim_{x\to\infty} \frac{e^x}{x^n}$ for any integer $n > 0$. Here top and bottom go to ∞ as x becomes very large, so the limit approaches $\frac{\infty}{\infty}$ and l'Hôpital applies; in fact it applies n times:

$$\lim_{x\to\infty} \frac{e^x}{x^n} \overset{\text{Hôp}}{=} \lim_{x\to\infty} \frac{e^x}{nx^{n-1}} \overset{\text{Hôp}}{=} \lim_{x\to\infty} \frac{e^x}{n(n-1)x^{n-2}} \overset{\text{Hôp}}{=} \cdots \overset{\text{Hôp}}{=} \lim_{x\to\infty} \frac{e^x}{n!x^0} = \infty\,,$$

since the top goes to ∞ and the bottom is the constant $n! = n(n-1)(n-2)\cdots(3)(2)(1)$. Another method: $f(z) = z^n$ is a continuous function, so we can pull it out of the limit.

$$\lim_{x\to\infty} \frac{e^x}{x^n} = \left(\lim_{x\to\infty} \frac{e^{x/n}}{x}\right)^n \overset{\text{Hôp}}{=} \left(\lim_{x\to\infty} \frac{\frac{1}{n}e^{x/n}}{1}\right)^n = (\infty)^n = \infty.$$

This result means that the exponential growth on the top is much faster than the polynomial growth on the bottom, so the quotient gets larger and larger along with x.

EXAMPLE: $\lim_{h\to 0^+} h^b e^{1/h^a}$ for any $a, b > 0$, of the form $0 \cdot \infty$. We need to simplify before L'Hôpital is any use. We substitute $x = 1/h^a$, $h = \frac{1}{x^{1/a}}$, so $x \to \infty$ as $h \to 0^+$. Then we pull out the power of x as in the previous example:

$$\lim_{h\to 0} h^b e^{1/h^a} = \lim_{x\to\infty} \frac{e^x}{x^{b/a}} = \left(\lim_{x\to\infty} \frac{e^{(a/b)x}}{x}\right)^{b/a} \overset{\text{Hôp}}{=} \infty^{b/a} = \infty.$$

EXAMPLE: Another $\frac{\infty}{\infty}$ form:

$$\lim_{x\to\infty} \frac{x^3 + x^2 + x + 1}{x^2 - x + 1} \overset{\text{Hôp}}{=} \lim_{x\to\infty} \frac{3x^2 + 2x + 1}{2x - 1} \overset{\text{Hôp}}{=} \lim_{x\to\infty} \frac{6x + 2}{2} = \infty$$

This means that the x^3 growth on top is much faster than the x^2 growth on the bottom. We can see this without L'Hôpital if we divide top and bottom by the smaller leading term, namely x^2:

$$\lim_{x\to\infty} \frac{\frac{1}{x^2}(x^3 + x^2 + x + 1)}{\frac{1}{x^2}(x^2 - x + 1)} = \lim_{x\to\infty} \frac{x + 1 + \frac{1}{x} + \frac{1}{x^2}}{1 - \frac{1}{x} + \frac{1}{x^2}}\,.$$

The top approaches $x + 1$ and the bottom approaches 1, so the quotient approaches ∞.

EXAMPLE: $\lim\limits_{x\to\infty} \ln(x) - x$, of indeterminate form $\infty - \infty$. We can wrangle up a quotient:

$$\lim_{x\to\infty} \ln(x) - x = \left(\lim_{x\to\infty} x\right)\left(\left(\lim_{x\to\infty} \tfrac{\ln(x)}{x}\right) - 1\right)$$

Since $\lim\limits_{x\to\infty} \frac{\ln(x)}{x} \overset{\text{Hôp}}{=} \lim\limits_{x\to\infty} \frac{1}{x} = 0$, the above becomes $\infty \cdot (0 - 1) = -\infty$.
Alternatively, $\lim\limits_{x\to\infty} \ln(x) - x = \ln\left(\lim\limits_{x\to\infty} \frac{x}{e^x}\right) \overset{\text{Hôp}}{=} \ln\left(\lim\limits_{x\to\infty} e^{-x}\right) = \ln(0^+) = -\infty$.

EXAMPLE: $\lim_{x\to\infty}(x+1)^p - x^p$ for $p>0$, a tough $\infty-\infty$ form. To create a quotient, we substitute $u=\frac{1}{x}\to 0^+$ in place of $x\to\infty$.

$$L = \lim_{x\to\infty}(x+1)^p - x^p \;=\; \lim_{u\to 0^+}(\tfrac{1}{u}+1)^p - (\tfrac{1}{u})^p \;=\; \lim_{u\to 0^+}\frac{(1+u)^p-1}{u^p} \;=\; \left(\lim_{u\to 0^+}\frac{((1+u)^p-1)^{1/p}}{u}\right)^p$$
$$\overset{\text{Hôp}}{=} \left(\lim_{u\to 0^+}\tfrac{1}{p}((1+u)^p-1)^{\frac{1}{p}-1}\cdot p(1+u)^{p-1}\right)^p \;=\; \left(\lim_{u\to 0^+}((1+u)^p-1)^{1-p}\right)\cdot\left(\lim_{u\to 0^+}(1+u)^{p-1}\right)^p$$

The second factor approaches 1, so the original limit is equal to the first factor, of the form $(0^+)^{1-p}$. This approaches $L=0$ if $p<1$; $L=1$ if $p=1$; and $L=\infty$ if $p>1$.

EXAMPLE: $\lim_{x\to a}\frac{\sqrt{f(x)-f(a)}}{f'(x)}$, where $f(x)$ has a non-degenerate minimum point, meaning $f'(a)=0$ but $f''(a)>0$ (concave up). Applying L'Hôpital to this $\frac{0}{0}$ limit:

$$L = \lim_{x\to a}\frac{\sqrt{f(x)-f(a)}}{f'(x)} \;\overset{\text{Hôp}}{=}\; \lim_{x\to a}\frac{\frac{f'(x)}{2\sqrt{f(x)-f(a)}}}{f''(x)} \;=\; \frac{1}{2f''(a)}\lim_{x\to a}\frac{f'(x)}{\sqrt{f(x)-f(a)}} \;=\; \frac{1}{2f''(a)}\cdot\frac{1}{L}\,.$$

Solving for L gives $L=\dfrac{1}{\sqrt{2f''(a)}}$. A simpler method is to pull out the radical:

$$\lim_{x\to a}\frac{\sqrt{f(x)-f(a)}}{f'(x)} \;=\; \sqrt{\lim_{x\to a}\frac{f(x)-f(a)}{(f'(x))^2}} \;\overset{\text{Hôp}}{=}\; \sqrt{\lim_{x\to a}\frac{f'(x)}{2f'(x)f''(x)}} \;=\; \frac{1}{\sqrt{2f''(a)}}.$$

Review Problem. Graph the function $f(x)=x^{1/x}$ for $x\geq 0$. First, the horizontal asymptote is $\lim_{x\to\infty}f(x)$ of indeterminate form $\infty^{1/\infty}=\infty^0$. By the Natural Base Principle (§6.4), $x^{1/x}=(e^{\ln(x)})^{1/x}=\exp(\frac{\ln(x)}{x})$, so:

$$\lim_{x\to\infty}x^{1/x} \;=\; \lim_{x\to\infty}\exp(\tfrac{\ln(x)}{x}) \;=\; \exp\Big(\lim_{x\to\infty}\tfrac{\ln(x)}{x}\Big)\,.$$

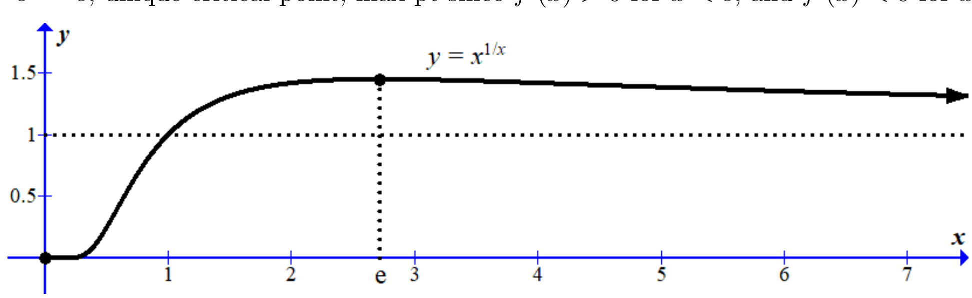
By L'Hopital $\lim_{x\to\infty}\frac{\ln(x)}{x}=\lim_{x\to\infty}\frac{1/x}{1}=0$, and the horizontal asymptote is $y=\exp(0)=1$.

On the other end, $f(0)$ is undefined but approaches $\lim_{x\to 0^+}x^{1/x} = \exp(\lim_{x\to 0^+}\frac{\ln(x)}{x})$. The inside limit is *not* indeterminate, rather $\frac{-\infty}{0^+}=-\infty$: large divided by tiny is very large. (Applying L'Hopital would give a *wrong answer*!) Thus set $f(0)=\lim_{x\to 0}f(x)=e^{-\infty}=0$.

The critical (max/min) points are where $f'(x)=0$. Using logarithmic differentiation on $\ln f(x)=\ln(x^{1/x})=(1/x)\ln(x)=\ln(x)/x$, we get:

$$f'(x) \;=\; f(x)\,(\ln f(x))' \;=\; f(x)\,(\tfrac{\ln(x)}{x})' \;=\; x^{\frac{1}{x}}\,\frac{\frac{1}{x}x-\ln(x)(1)}{x^2} \;=\; x^{\frac{1}{x}-2}(1-\ln(x)) \;=\; 0.$$

The first factor, being an exponential, can never be zero. The second factor gives $1-\ln(x)=0$, or $x=e^1=e$, unique critical point; max pt since $f'(x)>0$ for $x<e$, and $f'(x)<0$ for $x>e$.



CHALLENGES: Show $f'(0)=0$, $(f^{-1})'(\sqrt{2})=\frac{4}{\sqrt{2}(1-\ln(2))}$. Solve $n^m=m^n$ over whole numbers. Approximate inflection points by Newton's Method: $f''(x)=0$ at $x\approx 0.58193$ & 4.36777 .

Review problem: Graph $y=x^ne^{-x^2}$ for each $n\geq 1$ (moment functions of the Gaussian).