This technique evaluates limits which approach indeterminate forms like $0/0$ and $\infty/\infty$.

**Theorem:** For functions $f(x), g(x)$, suppose $f'(x), g'(x)$ exist and $g'(x) \neq 0$, on some interval $x \in (a-\delta, a+\delta)$. Suppose that either:

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0 \quad \text{or} \quad \lim_{x \to a} |f(x)| = \lim_{x \to a} |g(x)| = \infty.$$  

Then:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

provided the right side limit exists, or equals $\infty$ or $-\infty$.

There is another version for limits as $x$ becomes very large:

**Theorem:** Let $f(x), g(x)$ be functions which are differentiable and $g'(x) \neq 0$, on a semi-infinite interval $x \in (c, \infty)$. Suppose that either:

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} g(x) = 0 \quad \text{or} \quad \lim_{x \to \infty} |f(x)| = \lim_{x \to \infty} |g(x)| = \infty.$$  

Then:

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{f'(x)}{g'(x)},$$

provided the right side limit exists, or equals $\infty$ or $-\infty$.

The above also holds with $x \to \infty$ replaced with $x \to -\infty$.

**Proof.** There is an easy and enlightening proof of the Theorem if we assume:

$$\lim_{x \to a} f(x) = f(a) = 0, \quad \lim_{x \to a} g(x) = g(a) = 0,$$

$$\lim_{x \to a} f'(x) = f'(a), \quad \lim_{x \to a} g'(x) = g'(a) \neq 0.$$  

In this case:

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = \frac{f'(a)}{g'(a)} = \lim_{x \to a} \frac{f(x)-f(a)}{g(x)-g(a)} = \lim_{x \to a} \frac{f(x)-f(a)}{g(x)-g(a)} = \lim_{x \to a} \frac{f(x)}{g(x)}.$$  

That is, the quotient on the left is approximately $\frac{\Delta f}{\Delta g}$. But if $f$ starts at $f(a) = 0$, then the change in $f(x)$ is just the value of $f(x)$: that is, $\Delta f = f(x) - f(a) = f(x)$; and similarly $\Delta g = g(x)$.

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*A more complete proof. Assume only that $\lim_{x \to a} f(x) = f(a) = 0$, $\lim_{x \to a} g(x) = g(a) = 0$ and $\lim_{x \to a} f'(x)/g'(x)$ exists. This means $f'(x), g'(x)$ are defined and $g'(x) \neq 0$ near $x = a$. I claim that also $g(x) \neq 0$ near $x = a$. Otherwise, if we had $g(x) = 0$ arbitrarily near $x = a$, the Mean Value Theorem (§3.2) would imply $g'(c) = 0$ for $c \in (a, x)$ or $(x, a)$, contradicting the existence of $\lim_{x \to a} f'(x)/g'(x)$.

The Cauchy Mean Value Theorem (end of §3.2) says that if $f(x), g(x)$ are continuous on $[a, b]$, differentiable on $(a, b)$, then there is some $c \in (a, b)$ with $\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$, provided the denominators are non-zero. Applying this to any sufficiently small interval $[a, x]$ or $[x, a]$ gives some $c_c \in (a, x)$ or $(x, a)$ with $f(x)/g(x) = f'(c_c)/g'(c_c)$. Now, as $x \to a$, also $c_c \to a$, and $f(x)/g(x) = f'(c_c)/g'(c_c)$ clearly approaches the same value as $f'(x)/g'(x)$.
example: \( \lim_{x \to 2} \frac{x-2}{x^2-4} \). The top and bottom both approach zero, so the limit approaches the indeterminate form \( \frac{0}{0} \), and L'Hôpital’s Rule applies.

\[
\lim_{x \to 2} \frac{x-2}{x^2-4} \overset{\text{Hôp}}{=} \lim_{x \to 2} \frac{(x-2)'}{(x^2-4)'} = \lim_{x \to 2} \frac{1}{2x} = \frac{1}{4}.
\]

In this simple case, we can also find the limit by cancelling vanishing factors in the numerator and denominator:

\[
\lim_{x \to 2} \frac{x-2}{x^2-4} = \lim_{x \to 2} \frac{x-2}{(x-2)(x+2)} = \lim_{x \to 2} \frac{1}{x+2} = \frac{1}{4}.
\]

Similar reasoning would apply to the \( \frac{\infty}{\infty} \) form \( \lim_{x \to \infty} \frac{x-2}{x^2-4} \overset{\text{Hôp}}{=} \lim_{x \to \infty} \frac{1}{2x} = 0. \)

example: \( \lim_{x \to 0} \frac{e^x-1-x}{x^2} \). This approaches \( \frac{0}{0} \), so L'Hôpital applies.

\[
\lim_{x \to 0} \frac{e^x-1-x}{x^2} \overset{\text{Hôp}}{=} \lim_{x \to 0} \frac{e^x-0-1}{2x}.
\]

This still approaches \( \frac{0}{0} \), so we can use L'Hôpital again:

\[
\lim_{x \to 0} \frac{e^x-0-1}{2x} \overset{\text{Hôp}}{=} \lim_{x \to 0} \frac{e^x}{2} = \frac{1}{2}.
\]

example: \( \lim_{x \to 0^+} x \ln(x) \). (Here we use a one-sided limit \( x \to 0^+ \) because \( \ln(x) \) is undefined for \( x < 0 \).) This approaches the indeterminate form \( 0 \cdot (\infty) \), so it is a difficult limit, but we must manipulate it into a quotient to apply L'Hôpital:

\[
\lim_{x \to 0^+} x \ln(x) = \lim_{x \to 0^+} \frac{\ln(x)}{1/x}
\]

Now top and bottom become infinite approaching \( \frac{-\infty}{\infty} \), so L'Hôpital applies.

\[
\lim_{x \to 0^+} x \ln(x) = \lim_{x \to 0^+} \frac{\ln(x)}{1/x} = \lim_{x \to 0^+} \frac{1/x}{-1/x^2} = \lim_{x \to 0^+} (-x) = 0.
\]

example: \( \lim_{x \to 0} x^x \). This approaches the indeterminate form \( 0^0 \), but we can once again manipulate it into a limit we can handle:

\[
\lim_{x \to 0} x^x = \lim_{x \to 0} e^{\ln(x)x} = \lim_{x \to 0} \exp(x \ln(x)) = \exp \left( \lim_{x \to 0} x \ln(x) \right).
\]

We can move the limit inside \( \exp( \ ) \) because it is a continuous function (see §1.8 Composition Law). Applying the previous example, the limit becomes \( \exp(0) = 1 \).

example: \( \lim_{x \to 0} \frac{\sin(x)}{e^x} \). The bottom does not approach 0, so this is not indeterminate at all, and \( \text{L'Hôpital does not apply here} \). Instead, this is an easy limit that can be evaluated by continuity (plugging in):

\[
\lim_{x \to 0} \frac{\sin(x)}{e^x} = \frac{\sin(0)}{e^0} = \frac{0}{1} = 0.
\]

If we incorrectly try to apply L'Hôpital when it is not valid, we get a wrong answer:

\[
\lim_{x \to 0} \frac{\sin(x)}{e^x} = \lim_{x \to 0} \frac{\cos(x)}{e^x} = \frac{\cos(0)}{e^0} = 1 \text{ (WRONG)}.
\]
EXAMPLE: \( \lim_{x \to \infty} \frac{e^x}{x^n} \) for any integer \( n > 0 \). Here top and bottom go to \( \infty \) as \( x \) becomes very large, so the limit approaches \( \frac{\infty}{\infty} \) and \( \text{L'Hôpital} \) applies; in fact it applies \( n \) times:

\[
\lim_{x \to \infty} \frac{e^x}{x^n} = \lim_{x \to \infty} \frac{e^x}{n x^{n-1}} = \lim_{x \to \infty} \frac{e^x}{n(n-1)x^{n-2}} = \cdots = \lim_{x \to \infty} \frac{e^x}{n!x^0} = \infty,
\]

since the top goes to \( \infty \) and the bottom is the constant \( n! = n(n-1)(n-2) \cdots (3)(2)(1) \).

Another method: \( f(z) = z^n \) is a continuous function, so we can pull it out of the limit.

\[
\lim_{x \to \infty} \frac{e^x}{x^n} = \left( \lim_{x \to \infty} \frac{e^{x/n}}{x} \right)^n = \left( \lim_{x \to \infty} \frac{\frac{1}{n} e^{x/n}}{1} \right)^n = (\infty)^n = \infty.
\]

This result means that the exponential growth on the top is much faster than the polynomial growth on the bottom, so the quotient gets larger and larger along with \( x \).

EXAMPLE: Another \( \frac{\infty}{\infty} \) form:

\[
\lim_{x \to \infty} \frac{x^3 + x^2 + x + 1}{x^2 - x + 1} = \lim_{x \to \infty} \frac{3x^2 + 2x + 1}{2x - 1} = \lim_{x \to \infty} \frac{6x + 2}{2} = \infty
\]

This means that the \( x^3 \) growth on top is much faster than the \( x^2 \) growth on the bottom. We can see this without \( \text{L'Hôpital} \) if we divide top and bottom by the smaller leading term, namely \( x^2 \):

\[
\lim_{x \to \infty} \frac{\frac{1}{x^2}(x^3 + x^2 + x + 1)}{\frac{1}{x^2}(x^2 - x + 1)} = \lim_{x \to \infty} \frac{x + 1 + \frac{1}{x} + \frac{1}{x^2}}{1 - \frac{1}{x} + \frac{1}{x^2}}.
\]

The top approaches \( x + 1 \) and the bottom approaches 1, so the quotient approaches \( \infty \).

EXAMPLE: \( \lim_{x \to \infty} \ln(x) - x \), of indeterminate form \( \infty - \infty \). We can wrangle up a quotient:

\[
\lim_{x \to \infty} (\ln(x) - x) = \left( \lim_{x \to \infty} x \right) \left( \lim_{x \to \infty} \frac{\ln(x)}{x} - 1 \right)
\]

Since \( \lim_{x \to \infty} \frac{\ln(x)}{x} = \lim_{x \to \infty} \frac{1}{x} = 0 \), the above becomes \( \infty \cdot (0 - 1) = -\infty \).

Alternatively, \( \lim_{x \to \infty} \ln(x) - x = \ln \left( \lim_{x \to \infty} \frac{x}{e^x} \right) = \ln(0^+) = -\infty \).

EXAMPLE: \( \lim_{x \to \infty} (x+1)^p - x^p \) for \( p > 0 \), a tough \( \infty - \infty \) form. To create a quotient, we substitute \( u = \frac{1}{x} \to 0^+ \) in place of \( x \to \infty \).

\[
L = \lim_{x \to \infty} (x+1)^p - x^p = \lim_{u \to 0^+} \left( \frac{1}{u} + 1 \right)^p - \frac{1}{u}^p = \lim_{u \to 0^+} \frac{(1+u)^p - 1}{u^p} = \left( \lim_{u \to 0^+} \frac{(1+u)^p - 1}{u^p} \right)^p
\]

\[
= \left( \lim_{u \to 0^+} \frac{1}{p} \left( (1+u)^p - 1 \right)^{p-1} \cdot p(1+u)^{p-1} \right)^p = \left( \lim_{u \to 0^+} (1+u)^p - 1 \right)^{1-p} \cdot \left( \lim_{u \to 0^+} (1+u)^{p-1} \right)^p
\]

The second factor approaches 1, so the original limit is equal to the first factor, of the form \((0^+)^{1-p}\). This approaches \( L = 0 \) if \( p < 1 \); \( L = 1 \) if \( p = 1 \); and \( L = \infty \) if \( p > 1 \).
EXAMPLE: \( \lim_{x \to a} \frac{\sqrt{f(x) - f(a)}}{f'(x)} \), where \( f(x) \) has a non-stationary critical point, meaning \( f'(a) = 0 \) but \( f''(a) \neq 0 \). Applying L’Hospital to this \( \frac{0}{0} \) limit:

\[
L = \lim_{x \to a} \frac{\sqrt{f(x) - f(a)}}{f'(x)} \overset{Hôp}{=} \lim_{x \to a} \frac{\frac{f'(x)}{2\sqrt{f(x) - f(a)}}}{f''(x)} = \frac{1}{2f''(a)} \lim_{x \to a} \frac{f'(x)}{\sqrt{f(x) - f(a)}} = \frac{1}{2f''(a)} \cdot L.
\]

Solving for \( L \) gives \( L = \frac{1}{\sqrt{2f''(a)}} \). A simpler method is to pull out the radical:

\[
\lim_{x \to a} \frac{\sqrt{f(x) - f(a)}}{f'(x)} = \sqrt{\lim_{x \to a} \frac{f(x) - f(a)}{(f'(x))^2}} \overset{Hôp}{=} \sqrt{\lim_{x \to a} \frac{f'(x)}{2f'(x)f''(x)}} = \frac{1}{\sqrt{2f''(a)}}.
\]