Definitions. Besides the *algebraic functions* defined by arithmetic operations, constant powers, and roots, we have seen several types of *transcendental functions* such as $e^x$, the trigonometric functions, and their inverse functions. Now we introduce the *hyperbolic functions*, a new class of transcendental functions which appear in some scientific and mathematical applications (though much less commonly than our previous functions).

Each hyperbolic function corresponds to a trigonometric function: to the ordinary sine function $\sin(x)$ there corresponds the hyperbolic sine, written $\sinh(x)$; to the ordinary tangent there corresponds the hyperbolic tangent $\tanh(x)$, etc.* These new functions are defined in terms of exponential functions:

$$
\sinh(x) = \frac{e^x - e^{-x}}{2} \quad \cosh(x) = \frac{e^x + e^{-x}}{2} \quad \tanh(x) = \frac{\sinh(x)}{\cosh(x)}
$$

$$
\text{sech}(x) = \frac{1}{\cosh(x)} \quad \text{csch}(x) = \frac{1}{\sinh(x)} \quad \coth(x) = \frac{\cosh(x)}{\sinh(x)}
$$

That is, $\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$, etc. Graphs are easy to picture from $y = \frac{1}{2}e^x$ and $y = \frac{1}{2}e^{-x}$:

Notice that $\sinh(x)$ is an odd function like $\sin(x)$, meaning $f(-x) = -f(x)$; and $\cosh(x)$ is an even function like $\cos(x)$, meaning $f(-x) = f(x)$. Also, $e^x = \sinh(x) + \cosh(x)$, so the two primary hyperbolic functions are the odd and even components of the exponential function.†

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*We pronounce sinh as “sinch”, cosh as “kosh”, tanh as “tanch”, etc.

†The hyperbolic $e^x = \cosh(x) + \sinh(x)$ corresponds to Euler’s formula $e^{ix} = \cos(x) + i\sin(x)$, where $i = \sqrt{-1}$. Comparing, we find $\cosh(x) = \cos(ix)$ and $\sinh(x) = -i\sin(ix)$, which explains the analogy.
**Geometric meaning.** Why the trigonometric nomenclature? The most important geometric role of the trigonometric functions is to parametrize circular motion: \((x, y) = (\cos(t), \sin(t))\) traces out the unit circle for \(t \in [0, 2\pi]\). This is because the circle equation \(x^2 + y^2 = 1\) corresponds to the identity \(\cos^2(t) + \sin^2(t) = 1\).

It turns out the hyperbolic functions \((x, y) = (\cosh(t), \sinh(t))\) for \(t \in (-\infty, \infty)\) trace out a branch of the standard hyperbola defined by \(x^2 - y^2 = 1\), because of the basic hyperbolic identity \(\cosh^2(t) - \sinh^2(t) = 1\).

In fact, the shaded sector with corners \((0, 0), (1, 0), (\cosh(t), \sinh(t))\) has area \(\frac{1}{2}t\); just as in the circle, the sector with corners \((0, 0), (1, 0), (\cos(t), \sin(t))\) has area \(\frac{1}{2}t\).

The basic hyperbolic identity can easily be checked from the definitions:

\[
\cosh^2(x) - \sinh^2(x) = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2
\]

\[
= \left(e^{2x} + 2 + e^{-2x}\right) - \left(e^{2x} - 2 + e^{-2x}\right) = 4 = 1
\]

**Formulas.** The analogy goes much further: almost every formula involving trigonometric functions has a hyperbolic counterpart, often with changes in the ± signs.

\[
cosh^2(x) - \sinh^2(x) = \cos(x)
\]

\[
\sinh(x + y) = \sinh(x) \cosh(y) + \cosh(x) \sinh(y)
\]

\[
cosh(x + y) = \cosh(x) \cosh(y) + \sinh(x) \sinh(y)
\]

\[
\sinh'(x) = \cosh(x) \quad \cosh'(x) = \sinh(x)
\]

\[
\tanh'(x) = \text{sech}^2(x) \quad \text{sech}'(x) = -\tanh(x) \text{sech}(x)
\]

Each of these can be easily verified from the definitions via exponentials. For example:

\[
\sinh'(x) = \left(\frac{1}{2}(e^x - e^{-x})\right)' = \frac{1}{2}(e^x - (-e^{-x})) = \cosh(x).
\]

\[
\tanh'(x) = \left(\frac{\sinh(x)}{\cosh(x)}\right)' = \frac{\sinh'(x) \cosh(x) - \sinh(x) \cosh'(x)}{\cosh^2(x)}
\]

\[
= \frac{\cosh^2(x) - \sinh^2(x)}{\cosh^2(x)} = \frac{1}{\cosh^2(x)} = \text{sech}^2(x).
\]
EXAMPLE: If $\sinh(x) = -3$, find $\cosh(x)$. Solve $\cosh^2(x) - \sinh^2(x) = 1$ to get $\cosh(x) = \pm \sqrt{1 + (-3)^2} = \pm \sqrt{10}$; but $\cosh(x) = \frac{1}{2}(e^x + e^{-x}) > 0$ for all $x$, so $\cosh(x) = \sqrt{10}$.

EXAMPLE: Find the derivative of $\ln(\cosh(x))$. Using the Chain Rule:

$$[\ln(\cosh(x))]' = \frac{\cosh'(x)}{\cosh(x)} = \frac{1}{\cosh(x)} \cdot \sinh(x) = \tanh(x),$$

a signed analog of $[\ln(\cos(x))]' = [-\ln(\sec(x))]' = -\tan(x)$.

EXAMPLE: Find the antiderivative $\int \frac{\sinh(x)}{\cosh^2(x)} \, dx$. Substitute $u = \cosh(x)$, $du = \sinh(x) \, dx$:

$$\int \frac{\sinh(x)}{\cosh^2(x)} \, dx = \int \frac{1}{\cosh^2(x)} \sinh(x) \, dx = \int \frac{1}{u^2} \, du = -\frac{1}{u} = -\frac{1}{\cosh(x)} = -\text{sech}(x).$$

(For brevity, we neglect the arbitrary constant term $+C$.) Alternatively: $\int \frac{\sinh(x)}{\cosh^2(x)} \, dx = \int \tanh(x) \, \text{sech}(x) \, dx = -\text{sech}(x)$, directly from reversing our derivative table.

EXAMPLE: Find $\int \sinh^2(t) \, dt$. Exactly as for $\int \sin^2(\theta) \, d\theta$, use

$$\cosh(2t) = \cosh^2(t) + \sinh^2(t) = 2 \sinh^2(t) + 1,$$

so that $\sinh^2(t) = \frac{1}{2} \cosh(2t) - \frac{1}{2}$, and:

$$\int \sinh^2(t) \, dt = \int \left(\frac{1}{2} \cosh(2t) - \frac{1}{2}\right) \, dt = \frac{1}{2} \cosh(2t) - \frac{1}{2} t = \frac{1}{2} \cosh(t) \sinh(t) - \frac{1}{4} t.$$

EXAMPLE: Find $\int \text{sech}(x) \, dx$. The tricks for $\int \sec(x) \, dx$ do not work. Instead, write in terms of exponentials, and substitute $u = e^x$, $x = \ln(u)$, $dx = \frac{1}{u} \, du$:

$$\int \text{sech}(x) \, dx = \int \frac{2}{e^x + e^{-x}} \, dx = \int \frac{2e^x}{e^{2x} + 1} \, dx = \int \frac{2u}{u^2 + 1} \, \frac{1}{u} \, du = 2 \tan^{-1}(u) = 2 \tan^{-1}(e^x).$$

Inverse hyperbolic functions. We can define inverse hyperbolic functions and compute their derivatives just as for trig functions in §6.6, getting several more antiderivatives. For example, setting $y = \sinh(x)$, $x = \sinh^{-1}(y)$, we have $\cosh(x) = \sqrt{y^2 + 1}$ by the basic hyperbolic identity. We take $\frac{dy}{dx}$ of $y = \sinh(\sinh^{-1}(y))$ to find:

$$1 = \cosh(\sinh^{-1}(y)) (\sinh^{-1})'(y)$$

$$(\sinh^{-1})'(y) = \frac{1}{\cosh(\sinh^{-1}(y))} = \frac{1}{\cosh(x)} = \frac{1}{\sqrt{y^2 + 1}}.$$

Therefore:

$$\int \frac{1}{\sqrt{1 + y^2}} \, dy = \sinh^{-1}(y) + C.$$

We can get a more elementary form for $x = \sinh^{-1}(y)$ by solving the equation:

$$y = \sinh(x) = \frac{1}{2} (e^x - e^{-x}) = \frac{1}{2} (e^x - e^{-x}) = \frac{(e^x)^2 - 1}{2e^x}.$$

That is, $(e^x)^2 - 2y(e^x) - 1 = 0$, so the Quadratic Formula gives $e^x = \frac{1}{2} (2y \pm \sqrt{4y^2 + 4})$,

$$\sinh^{-1}(y) = x = \ln(y \pm \sqrt{y^2 + 1}).$$

Here $\pm$ must be $+$ since the input of logarithm must be positive.
We summarize a number of similar formulas, omitting $+C$.

\[
\int \frac{1}{\sqrt{x^2 - 1}} \, dx = \cosh^{-1}(x) = \ln(x + \sqrt{x^2 - 1})
\]

\[
\int \frac{1}{\sqrt{x^2 + 1}} \, dx = \sinh^{-1}(x) = \ln(x + \sqrt{x^2 + 1})
\]

\[
\int \frac{-1}{x\sqrt{1 - x^2}} \, dx = \text{sech}^{-1}(x) = -\ln(x) + \ln(1 + \sqrt{1 - x^2})
\]

\[
\int \frac{-1}{x\sqrt{1 + x^2}} \, dx = \text{csch}^{-1}(x) = -\ln(x) + \ln(1 + \sqrt{1 + x^2})
\]

\[
\int \frac{1}{1 - x^2} \, dx = \tanh^{-1}(x) = \frac{1}{2} \ln(1 + x) - \frac{1}{2} \ln(1 - x)
\]

We sometimes denote $\sin^{-1}$ as arcsin, and we may denote $\sinh^{-1}$ as arsinh or arsh, etc.

**Example:** Integrate $\int \frac{1}{\sqrt{x^2 + x}} \, dx$. We want to manipulate this into one of the above forms. Completing the square, we have

\[
x^2 + x = x^2 + 2\left(\frac{1}{2}\right)x + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 = (x + \frac{1}{2})^2 - \frac{1}{4} = \frac{1}{4}(2x + 1)^2 - 1,
\]

so the substitution $u = 2x + 1$, $du = 2 \, dx$ gives:

\[
\int \frac{1}{\sqrt{x^2 + x}} \, dx = \int \frac{1}{\sqrt{(2x+1)^2 - 1}} \, 2 \, dx = \int \frac{1}{\sqrt{u^2 - 1}} \, du
\]

\[
= \cosh^{-1}(u) = \cosh^{-1}(2x + 1) = \ln\left(2x + 1 + 2\sqrt{x^2 + x}\right).
\]