Exponential Growth and Decay

Differential equations. An algebra equation involves a variable representing an unknown number, often denoted by x; and to solve the equation means to find the numerical values of x which make the equation true. A *differential equation* (DE) involves an *unknown function*, often $y = f(x)$, and its derivatives $\frac{dy}{dx} = f'(x)$, $\frac{d^2y}{dx^2} = f''(x)$, etc. To solve the DE means to find the explicit functions $f(x)$ which make the equation true. For example, the differential equation $f'(x) = 2x$ has solution functions $f(x) = x^2 + C$ for any constant C.

Scientific laws attempt to give simple explanations of complicated phenomena. Some, such as the principle of evolution through natural selection, are *qualitative* laws, stated in ordinary language. Those laws which are *quantitative*, precisely explaining numerical measurements, are usually stated in terms of simple differential equations: the complication arises from the solutions of these equations. The theory of differential equations is one of the richest, most extensive fields of mathematics: in fact, the most important equations, such as those describing fluid flow, are each large areas of study all by themselves. Here we will consider only a few very elementary, easy-to-solve examples.

Equations solved by immediate integration. The very simplest DE's are those of the form $f'(x) = a(x)$, where $y = f(x)$ is the unknown and $a(x)$ is some given, known function. The solution is just the indefinite integral (anti-derivative) with $A'(x) = a(x)$:

$$f'(x) = a(x) \quad \Longrightarrow \quad f(x) = \int a(x)\,dx = A(x) + C\,.$$

Here $A(x)$ is found by reversing the derivative rules; but if this is not possible, we can always write $A(x) = \int_c^x a(t)\,dt$, a definite integral which can be approximated by Riemann sums. The constant C is often determined by an intitial condition on $f(0)$.

EXAMPLE: Suppose your car starts at a standstill, and you press the accelerator very slowly so that after 1 second you are gaining 1 mph each sec, after 2 seconds you are gaining 2 mph each sec, etc. How far have you traveled in t seconds, and how many seconds until you travel 1000 ft?

Here the unknown function $y = f(t)$ is the distance traveled (the position past the starting point) in feet; the velocity is $\frac{dy}{dt} = f'(t)$; and the acceleration $\frac{d^2y}{dt^2} = f''(t)$ is given as t mph per sec. Converting to consistent units, 1 mph $= \frac{5280 \text{ ft}}{3600 \text{ sec}} \approx 1.5$ ft/sec, so 1 mph per sec ≈ 1.5 ft/sec^2. Thus:

$$f''(t) \;=\; 1.5\,t, \quad \text{initial conditions } f(0) = 0,\ f'(0) = 0.$$

Initial distance traveled is zero; initial velocity is zero because we start at a standstill. We call this a *second order* differential equation because it involves the second derivative $f''(x)$. To solve, we integrate twice:

$$f'(t) \;=\; \int 1.5\,t\,dt \;=\; 0.75\,t^2 + C_1\,, \qquad f(t) \;=\; \int 0.75\,t^2 + C_1\;dt \;=\; 0.25\,t^3 + C_1 t + C_2\,.$$

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As usual with second-order equations, we obtain a family of solutions with two arbitrary constants C_1, C_2. (First-order quations with only $f'(t)$ will have only one constant in their solution.) The initial conditions determine the constants:

$$0 = f'(0) = 0.75(0^2) + C_1 = C_1 \quad \text{and} \quad 0 = f(0) = 0.25(0^3) + C_1(0) + C_2 = C_2\,.$$

Therefore: $f(t) = 0.25\,t^3$. Finally, solving $f(t) = 0.25\,t^3 = 1000$ answers the original question: it takes $t = \sqrt[3]{4000} \approx 16$ sec to travel 1000 ft.

Exponential growth from self-reproduction. The three qualitative mechanisms of evolution are *self-reproduction*; *variation* via genetic mutation and sexual recombination; and *selection* for reproductive capacity. To quantify self-reproduction, we first consider the simple situation of constant fertility without constraints: on average, each individual produces a certain number of offspring per unit time. Thus, if $P(t)$ is the population at time t, and k is the observed constant reproduction rate per individual, we get the DE:

$$\frac{dP}{dt} \;=\; kP\,.$$

To predict the population, we must solve for the unknown function $P(t)$. This time, integrating both sides will not help, since the right side is just as unknown as the left side. However, it is easy to guess a solution function: $P(t) = e^{kt}$, with $P'(t) = e^{kt} \cdot (kt)' = P(t)\,k$. Notice that the solution (an exponential function with the weird constant e) is considerably more complicated than the original equation.

Is this the only solution, the only population function consistent with the equation? In fact, any multiple of this will also work:

$$P(t) = ce^{kt}.$$

I claim that this is the most general solution of the differential equation. In fact, if $P(t)$ is any solution function with $P'(t) = k\,P(t)$, then the Quotient Rule (§2.3) says:

$$\left(\frac{P(t)}{e^{kt}}\right)' \;=\; \frac{P'(t)e^{kt} - P(t)(e^{kt})'}{(e^{kt})^2} \;=\; \frac{k\,P(t)\,e^{kt} - P(t)\,k\,e^{kt}}{e^{2kt}} \;=\; 0.$$

Since $\frac{P(t)}{e^{kt}}$ has zero derivative, it must be a constant function (§3.2):

$$\left(\frac{P(t)}{e^{kt}}\right)' \;=\; 0 \qquad \Longrightarrow \qquad \frac{P(t)}{e^{kt}} \;=\; c \qquad \Longrightarrow \qquad P(t) \;=\; c\,e^{kt}\,.$$

That is, any solution $P(t)$ must have the desired form.

Exponential growth is so explosive that any self-reproducing population will very quickly fill its environment, no matter how large. For example, if we started with a single cell one micrometer across and repeatedly doubled it every hour, we would fill the entire volume of the ocean (about $10^{18}\,\mathrm{m}^3$) within 120 generations, or 5 days. That is, as soon as a new trait produces a selective advantage which allows exponential reproduction even at a low rate, it will almost immediately populate all the livable space, at which point growth must stop for lack of resources (the net fertility rate k drops to zero).

Conversely, wherever we find exponential growth, we expect it to be caused by self-reproduction. For example Moore's Law predicts that the amount of computing power available at a fixed price will double every 18 months. This is possible because the key tools needed to design and manufacture better computers are our current computers.

Exponential growth doubling problem. Here is a common type of problem which needs no calculus, apart from the general exponential formula found above. Let $P(t)$ grams be the population of bacteria in a tank at t hours. Suppose the population doubles every 3 hours, and $P(1) = 2$. Find $P(t)$.

We must translate all the words of the problem (physical level) into equations (algebraic level). First, doubling in constant time means exponential growth: $P(t) = c\,e^{kt}$; but here it is easier to write $a = e^k$, so that $P(t) = ca^t$. We need only find the unknown constants c, a. For a given population $P(t)$, the population 3 hours later will be twice as much:

$$P(t{+}3) = 2P(t) \quad \Longrightarrow \quad c\,a^{t+3} = 2c\,a^t \quad \Longrightarrow \quad a^t a^3 = 2a^t \quad \Longrightarrow \quad a = \sqrt[3]{2} \approx 1.26\ .$$

The initial condition becomes: $P(1) = c\,a^1 = 2$, so that $c = 2/a = 2/\sqrt[3]{2} = 2^{2/3} \approx 1.59$.

$$P(t) \;=\; 2^{2/3}\,2^{(1/3)t} \;=\; 2^{(t+2)/3}.$$

Beware: any exponential model will break down when the population outgrows its available resources. After that time, our prediction is invalid.

The reciprocal of exponential growth is *exponential decay*: a process in which, instead of doubling in constant time, a quantity shrinks by half, meaning $k < 0$ or $a = e^k < 1$.

Separation of variables method. Certain easy DE's can be reduced to an integration problem by a simple trick. For some given functions $a(x), b(x)$, we want to find the unknown $f(x)$ satisfying:

$$f'(x) \;=\; a(x)\,b(f(x))\,.$$

Here the second factor is an expression in the unknown $f(x)$. Moving $f(x)$ to the left side, and taking the integral of both sides gives:

$$\int \frac{1}{b(f(x))}\,f'(x)\,dx \;=\; \int a(x)\,dx.$$

The left-hand integral can be simplified by the substitution $y = f(x)$, $dy = f'(x)\,dx$, giving $\int \frac{1}{b(y)}\,dy$. (Here $f(x)$ is unknown, but it is *some* function, so it can be used in a substitution.) Assuming we can find antiderivatives $\int \frac{1}{b(y)}\,dy = B(y)$ and $\int a(x)\,dx = A(x) + C$, our equation becomes:

$$\int \frac{1}{b(y)}\,dy = \int a(x)\,dx \quad \Longrightarrow \quad B(y) = B(f(x)) = A(x) + C \quad \Longrightarrow \quad f(x) = B^{-1}(A(x){+}C)\,,$$

assuming we can find an inverse function B^{-1} to solve $B(y) = A(x) + C$.

This reasoning looks especially natural in Leibniz notation, letting $y = f(x)$:

$$\frac{dy}{dx} = a(x)\,b(y) \quad \Longrightarrow \quad \frac{1}{b(y)}\,\frac{dy}{dx} = a(x) \quad \Longrightarrow \quad \int \frac{1}{b(y)}\,\frac{dy}{dx}\,dx = \int a(x)\,dx$$

$$\Longrightarrow \quad \int \frac{1}{b(y)}\,dy = \int a(x)\,dx \quad \Longrightarrow \quad B(y) = A(x) + C \quad \Longrightarrow \quad y \;=\; B^{-1}(A(x){+}C).$$

EXAMPLE: Applying the method to the exponential growth equation from before:

$$\frac{dP}{dt} = kP \quad \Longrightarrow \quad \int \frac{1}{P}\,\frac{dP}{dt}\,dt = \int k\,dt \quad \Longrightarrow \quad \int \frac{1}{P}\,dP = \int k\,dt$$

$$\Longrightarrow \quad \log|P| = kt + C \quad \Longrightarrow \quad P = \pm e^{kt+C} = \pm e^{C} e^{kt}\,.$$

This is our previous answer, except the arbtrary coefficient is $\pm e^C$ instead of c.*

EXAMPLE: Solve $f'(x) = \frac{\sin(x)}{f(x)}$. That is, the derivative of the unknown function $y = f(x)$ must be equal to $\sin(x)$ divided by $f(x)$ itself. The method gives:

$$\frac{dy}{dx} = \frac{\sin(x)}{y} \quad \Longrightarrow \quad \int y\,\frac{dy}{dx}\,dx = \int \sin(x)\,dx \quad \Longrightarrow \quad \int y\,dy = \int \sin(x)\,dx$$

$$\Longrightarrow \quad \tfrac{1}{2}y^2 = -\cos(x) + C \quad \Longrightarrow \quad y = \pm\sqrt{2C - 2\cos(x)}\,.$$

We can check our solution by plugging it into the original differential equation. Indeed, by the Chain Rule, $f'(x) = \frac{1}{2}(2C - 2\cos(x))^{-1/2} \cdot 2\sin(x) = \frac{\sin(x)}{f(x)}$.

Newton's Law of Cooling. This is a toy example of how scientific laws are expressed by DE's. Newton proposed a simple rule for the temperature $T(t)$ of a hot body as it cools down to a constant environmental temperature E: the rate of cooling is proportional to the difference between the body's temperature and the environment. The DE is:

$$\frac{dT}{dt} \;=\; -k\,(T - E)\,.$$

Here $k > 0$, so a high temperature cools quickly. Separating variables gives:

$$\int \frac{1}{T-E}\,\frac{dT}{dt}\,dt \;=\; -\int k\,dt \quad \Longrightarrow \quad \int \frac{1}{T-E}\,dT \;=\; -\int k\,dt$$

$$\Longrightarrow \quad \ln|T-E| = -kt + C \quad \Longrightarrow \quad T = E \pm e^{C} e^{-kt} = E + ce^{-kt}.$$

That is, $T(t)$ approaches the horizontal asymptote E by exponential decay.

EXAMPLE: In a room at 20°C, a cup of boiling tea (100°C) cools to 80°C in 1 minute. How long until it is sippable at 50°C? To answer, we assume $T(t) = E + ce^{-kt} = 20 + ce^{-kt}$ according to Newton's Law. The two intial conditions determine the parameters:

$$T(0) = 100 = 20 + c \quad \Longrightarrow \quad c = 80.$$

$$T(1) = 80 = 20 + 80e^{-k} \quad \Longrightarrow \quad e^{-k} = \tfrac{60}{80} = \tfrac{3}{4}\,.$$

Thus $T(t) = 20 + 80\big(\frac{3}{4}\big)^t$, and solving $T(t) = 50$ gives $t \approx 3.4$ min.

A few more examples of differential equations are solved in the *Synthesis* lecture notes.

*The form of the constant is irrelevant. For example, if we had put constants in both general antiderivatives, $\log|P| + C_1 = kt + C_2$, it would lead to $P = \pm e^{C_1 - C_2} e^{kt}$, where C_1, C_2 are arbitrary constants. But this does not give a more general solution than $\pm e^C$ for a single arbitrary C, or just a simple constant coefficient c.

First order linear DE. Another general method solves differential equations of type:

$$y'(x) \ = \ a(x)y(x) + b(x),$$

for given $a(x), b(x)$. The corresponding *homgeneous* equation, with unknown $z(x)$, is:

$$z'(x) \ = \ a(x)z(x) \quad \Longrightarrow \quad \int \frac{1}{z}\, dz \ = \ \int a(x)\, dx \quad \Longrightarrow \quad \ln|z| \ = \ A(x) + C \quad \Longrightarrow \quad z \ = \ ce^{A(x)},$$

where $A(x) = \int a(x)\, dx$, and $c = \pm e^C$ is an arbitrary constant.

We can solve the original equation $y' = ay + b$ using the trick of writing a solution in the form $y(x) = z(x)\, m(x)$, where $z(x) = e^{A(x)}$ is a solution of $z' = az$, and $m(x)$ is an unknown function known as the *integrating factor*. Then:

$$m' \ = \ (y/z)' = \frac{y'z - yz'}{z^2} \ = \ \frac{(ay+b)z - y\, az}{z^2} \ = \ b/z \ = \ b/e^{A(x)} \quad \Longrightarrow \quad m \ = \ \int \frac{b(x)}{e^{A(x)}}\, dx + C.$$

$$y(x) \ = \ z(x)\, m(x) \ = \ e^{A(x)} \left(\int \frac{b(x)}{e^{A(x)}}\, dx + C \right).$$

In fact, this $y(x)$ is the most general solution. Indeed, if $\tilde{y}(x)$ is an arbitrary solution, we have $(\tilde{y} - y)' = (a\tilde{y} + b) - (ay + b) = a(\tilde{y} - y)$; that is, $\tilde{y} - y$ satisfies the homogeneous equation, so $\tilde{y} - y = \tilde{C}e^{A(x)}$ and $\tilde{y} = y + \tilde{C}e^{A(x)}$, which is included in the family $y(x)$.