Math 133 # General Exp and Log **Stewart §6.4**

Derivative of general exp. To compute with functions of arbitrary base, we will repeatedly apply:

> *Natural Base Principle:* To deal with general exponentials and logarithms in calculus, write them in terms of the natural base e functions e^x and $\ln(x)$, which have $(e^x)' = e^x$ and $\ln'(x) = \frac{1}{x}$.

For example, we have $a = e^{\ln(a)}$, so:

$$(a^x)' \;=\; (e^{\ln(a)\,x})' \;=\; \exp'(\ln(a)\,x)\cdot(\ln(a)\,x)' \;=\; e^{\ln(a)\,x}\cdot\ln(a) \;=\; \ln(a)\,a^x.$$

Note that one factor is just our original function a^x, because differentiating the outside function e^x has no effect. In the other factor, $\ln(a)$ is a (complicated) constant, so $(\ln(a)\,x)' = \ln(a)$.

Derivative of general log. Since $f(x) = a^x = e^{\ln(a)\,x}$, we can find the inverse function $f^{-1}(y) = \log_a(y)$ by solving $y = e^{\ln(a)\,x}$ to get: $\ln(y) = \ln(a)\,x$, and $x = \frac{\ln(y)}{\ln(a)}$. That is, $f^{-1}(y) = \log_a(y) = \frac{\ln(y)}{\ln(a)}$. Switching the input variable to x, we get the *logarithm base change formula*:

$$\log_a(x) = \frac{\ln(x)}{\ln(a)}\,.$$

Hence:

$$\log_a'(x) = \left(\frac{\ln(x)}{\ln(a)}\right)' = \frac{1}{\ln(a)}\,\ln'(x) = \frac{1}{\ln(a)\,x}\,.$$

Problems.

EXAMPLE: Differentiate $f(x) = 6^{x+\cos(x)}$. It is *not* helpful to factor: $f(x) = 6^x 6^{\cos(x)}$. Instead, we have $6 = e^{\ln(6)}$, so:

$$\begin{aligned} f'(x) &= \left(e^{\ln(6)(x+\cos(x))}\right)' \\ &= \exp'(\ln(6)(x+\cos(x))\cdot\ln(6)(x+\cos(x))' \\ &= e^{\ln(6)(x+\cos(x))}\cdot\ln(6)\,(1-\sin(x)) \\ &= 6^{x+\cos(x)}\ln(6)\,(1-\sin(x)) \end{aligned}$$

Notice that the original function is again a factor of the derivative, because the derivative of the outside exp is itself.

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EXAMPLE: Differentiate $f(x) = x^x$. Since $x = e^{\ln(x)}$, we have:

$$\begin{aligned} f'(x) &= \left(e^{\ln(x)\,x}\right)' \\ &= \exp'(\ln(x)\,x) \cdot (\ln(x)\,x)' \\ &= \exp'(\ln(x)\,x) \cdot (\ln'(x)\,x + \ln(x)\,x') \\ &= x^x\,(1+\ln(x))\,. \end{aligned}$$

Once again, the original function is a factor of the derivative.
Another approach is the logarithmic derivative, based on the formula:

$$(\ln(f(x)))' \;=\; \ln'(f(x))\,f'(x) \;=\; \frac{f'(x)}{f(x)} \quad\Longrightarrow\quad f'(x) = f(x)\,(\ln(f(x)))'.$$

For our function, $\ln(f(x)) = \ln(x^x) = x\,\ln(x)$, and we quickly get the previous answer:

$$f'(x) \;=\; f(x)\,(\ln(f(x)))' \;=\; x^x(x\,\ln(x))' \;=\; x^x(1+\ln(x)).$$

EXAMPLE: Find the indefinite integral* $\int x\,6^{x^2}\,dx$.

We write in terms of natural functions, and do the substitution $u = \ln(6)\,x^2$:

$$\begin{aligned} \int x\,6^{x^2}\,dx \;&=\; \int x\,e^{\ln(6)x^2}dx \;=\; \frac{1}{2\ln(6)}\int e^{\ln(6)x^2}\;\ln(6)\,2x\,dx \\ &=\; \frac{1}{2\ln(6)}\int e^u\,du \;=\; \frac{e^u}{2\ln(6)} \;=\; \frac{e^{\ln(6)x^2}}{2\ln(6)} \;=\; \frac{6^{x^2}}{2\ln(6)} \end{aligned}$$

*The notation $\int f(x)\,dx$, with no limits of integration, is simply a shorthand for the general antiderivative, and is called the *indefinite integral*. Indeed, if we find the indefinite integral $\int f(x)\,dx = F(x) + C$, where $F'(x) = f(x)$, then we can evaluate the definite integral: $\int_a^b f(x)\,dx = [F(x)]_{x=a}^{x=b}$.