Derivative of general exp. To compute with functions of arbitrary base, we will repeatedly apply:

**Natural Base Principle:** To deal with general exponentials and logarithms in calculus, write them in terms of the natural base $e$ functions $e^x$ and $\ln(x)$, which have $(e^x)' = e^x$ and $\ln'(x) = \frac{1}{x}$.

For example, we have $a = e^{\ln(a)}$, so:

$$(a^x)' = (e^{\ln(a)\cdot x})' = \exp'(\ln(a)\cdot x) \cdot (\ln(a)\cdot x)' = e^{\ln(a)\cdot x} \cdot \ln(a) = \ln(a) a^x.$$ 

Note that one factor is just our original function $a^x$, because differentiating the outside function $e^x$ has no effect. In the other factor, $\ln(a)$ is a (complicated) constant, so $(\ln(a)\cdot x)' = \ln(a)$.

**Derivative of general log.** Since $f(x) = a^x = e^{\ln(a)\cdot x}$, we can find the inverse function $f^{-1}(y) = \log_a(y)$ by solving $y = e^{\ln(a)\cdot x}$ to get: $\ln(y) = \ln(a)\cdot x$, and $x = \frac{\ln(y)}{\ln(a)}$. That is, $f^{-1}(y) = \log_a(y) = \frac{\ln(y)}{\ln(a)}$. Switching the input variable to $x$, we get the logarithm base change formula:

$$\log_a(x) = \frac{\ln(x)}{\ln(a)}.$$ 

Hence:

$$\log'_a(x) = \left(\frac{\ln(x)}{\ln(a)}\right)' = \frac{1}{\ln(a)} \ln'(x) = \frac{1}{\ln(a) x}.$$ 

**Problems.**

**Example:** Differentiate $f(x) = 6^x + \cos(x)$. It is *not* helpful to factor: $f(x) = 6^x 6^{\cos(x)}$. Instead, we have $6 = e^{\ln(6)}$, so:

$$f'(x) = \left(e^{\ln(6)(x+\cos(x))}\right)'$$

$$= \exp'(\ln(6)(x+\cos(x)) \cdot \ln(6)(x+\cos(x))')$$

$$= e^{\ln(6)(x+\cos(x))} \cdot \ln(6) (1 - \sin(x))$$

$$= 6^{x+\cos(x)} \ln(6) (1 - \sin(x))$$

Notice that the original function is again a factor of the derivative, because the derivative of the outside exp is itself.

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**EXAMPLE:** Differentiate \( f(x) = x^x \). Since \( x = e^{\ln(x)} \), we have:

\[
\begin{align*}
f'(x) &= (e^{\ln(x)}x)' \\
&= \exp'(\ln(x)x) \cdot (\ln(x)x)' \\
&= \exp'(\ln(x)x) \cdot (\ln'(x)x + \ln(x)x') \\
&= x^x(1 + \ln(x)).
\end{align*}
\]

Once again, the original function is a factor of the derivative.

Another approach is the logarithmic derivative, based on the formula:

\[
(\ln(f(x)))' = \ln'(f(x)) f'(x) = \frac{f'(x)}{f(x)} \implies f'(x) = f(x)(\ln(f(x))').
\]

For our function, \( \ln(f(x)) = \ln(x^x) = x \ln(x) \), and we quickly get the previous answer:

\[
f'(x) = f(x)(\ln(f(x)))' = x^x(x \ln(x))' = x^x(1 + \ln(x)).
\]

**EXAMPLE:** Find the indefinite integral\(^*\) \( \int x \, 6x^2 \, dx \).

We write in terms of natural functions, and do the substitution \( u = \ln(6) x^2 \):

\[
\int x \, 6x^2 \, dx = \int x \, e^{\ln(6)x^2} \, dx = \frac{1}{2 \ln(6)} \int e^{\ln(6)x^2} \, \ln(6) \, 2x \, dx
\]

\[
= \frac{1}{2 \ln(6)} \int e^u \, du = \frac{e^u}{2 \ln(6)} = \frac{e^{\ln(6)x^2}}{2 \ln(6)} = \frac{6x^2}{2 \ln(6)}
\]

\(^*\)The notation \( \int f(x) \, dx \), with no limits of integration, is simply a shorthand for the general antiderivative, and is called the indefinite integral. Indeed, if we find the indefinite integral \( \int f(x) \, dx = F(x) + C \), where \( F'(x) = f(x) \), then we can evaluate the definite integral: \( \int_a^b f(x) \, dx = [F(x)]_{x=a}^{x=b} \).