

Basic Properties. Here is pretty much all you need to know about the natural exponential function $\exp(x)$ and the natural logarithm $\ln(x)$.

- $\exp(x) = e^x$ $\ln(x) = \log_e(x)$
- $e^{\ln(x)} = x$ $\ln(e^x) = x$ $e^a = b \iff a = \ln(b)$
- $e^0 = 1$ $e^1 = e \approx 2.718$ $\ln(1) = 0$ $\ln(e) = 1$
- $e^{x_1} e^{x_2} = e^{x_1+x_2}$ $e^{x_1}/e^{x_2} = e^{x_1-x_2}$ $(e^x)^p = e^{px}$
- $\ln(x_1 x_2) = \ln(x_1) + \ln(x_2)$ $\ln(x_1/x_2) = \ln(x_1) - \ln(x_2)$ $\ln(x^p) = p \ln(x)$
- $(e^x)' = e^x$ $\int e^x dx = e^x + C$ $\ln'(x) = \frac{1}{x}$ $\int \frac{1}{x} dx = \ln|x| + C$

We give some tricky examples, applying the basic facts and the Chain Rule.

EXAMPLE: Solve for y in the equation: $\ln(ye^x) + 1 = 2x + \ln(y^2)$.

Strategy: Expand into a sum, move y 's to the left, all else to the right.

$$\ln(y) + \ln(e^x) + 1 = 2x + 2 \ln(y)$$

$$\ln(y) - 2 \ln(y) = 2x - x - 1$$

$$\ln(y) = 1 - x$$

$$y = e^{1-x}.$$

EXAMPLE: Differentiate $f(x) = \sin(e^{\tan(x)})$.

Strategy: Apply the Chain Rule with outer = $\sin(x)$, inner = $e^{\tan(x)}$.

$$\begin{aligned} (\sin(e^{\tan(x)}))' &= \sin'(e^{\tan(x)}) \cdot (e^{\tan(x)})' \\ &= \cos(e^{\tan(x)}) \cdot \exp'(\tan(x)) \cdot \tan'(x) \\ &= \cos(e^{\tan(x)}) \cdot e^{\tan(x)} \cdot \sec^2(x) \end{aligned}$$

EXAMPLE: Differentiate $f(x) = \int_2^{e^x} \ln(t \sin(t)) dt$.

Strategy: Apply the Chain Rule with outer function $I(x) = \int_2^x \ln(t \sin(t)) dt$. The First Fundamental Theorem (§4.3) says* that $g'(x) = \ln(x \sin(x))$. We are given a composition of functions $f(x) = g(e^x)$, so the Chain Rule applies:

$$f'(x) = g'(e^x) \cdot (e^x)' = \ln(e^x \sin(e^x)) \cdot e^x = xe^x + \ln(\sin(e^x)) e^x.$$

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*That is, in the plane with t and y axes, the integral function $I(x)$ is the area between the curve $y = \ln(t \sin(t))$ and the interval $t \in [1, x]$. The rate of change area as the interval expands at $t = x$ is equal to the height of the curve at the moving end: $I'(x) = \ln(x \sin(x))$.

Algebraically, the derivative of the integral of a function gives back the original function.

EXAMPLE: Find $\frac{dy}{dx}$ by implicit differentiation, for (x, y) satisfying:

$$e^y = \cos(x+y).$$

Specifically, find $\frac{dy}{dx}$ at the point $(x, y) = (0, 0)$.

Strategy: The equation defines some unknown curve containing the point $(x, y) = (0, 0)$, since $e^0 = \cos(0+0)$. We want the slope of the tangent line at that point. Assuming $y = y(x)$ is some function which satisfies the equation, we apply the Chain Rule to both sides, and solve for $y' = \frac{dy}{dx}$.

$$\begin{aligned}(e^{y(x)})' &= \cos(x + y(x))' \\ \exp'(y(x)) \cdot y'(x) &= \cos'(x + y(x)) \cdot (x + y(x))' \\ e^y \cdot y' &= -\sin(x+y) \cdot (1+y') \\ (e^y + \sin(x+y)) y' &= -\sin(x+y) \\ y' &= -\frac{\sin(x+y)}{e^y + \sin(x+y)}.\end{aligned}$$

Substituting $(x, y) = (0, 0)$ gives $y' = \left. \frac{dy}{dx} \right|_{x=0} = -\frac{\sin(0+0)}{e^0 + \sin(0+0)} = 0$. That is, the unknown curve has a horizontal tangent at the origin.

EXAMPLE: Find the derivative of $f(x) = a^x$ for any base $a > 0$.

Strategy: write a^x in terms of the natural exponential, whose derivative is known. Specifically, solving $a = e^p$ by $p = \ln(a)$, we get $a = e^{\ln(a)}$, and $a^x = (e^{\ln(a)})^x = e^{\ln(a)x}$. Applying the Chain Rule:

$$\begin{aligned}(a^x)' &= (e^{\ln(a)x})' = \exp'(\ln(a)x) \cdot (\ln(a)x)' \\ &= e^{\ln(a)x} \cdot \ln(a) = \ln(a) a^x.\end{aligned}$$

Note that $\ln(a)$ is a constant, so $(\ln(a)x)' = \ln(a)$.[†]

[†]If we tried to apply the Product and Chain Rules, we would get:

$$(\ln(a)x)' = (\ln(a))' \cdot x + \ln(a) \cdot (x)' = \ln'(a) \cdot a' \cdot x + \ln(a) \cdot 1 = \frac{1}{a} \cdot 0 \cdot x + \ln(a) = \ln(a).$$