Review of exponential and logarithm functions. We recall some facts from algebra, which we will later prove from a calculus point of view. In an expression of the form $a^p$, the number $a$ is called the base and the power $p$ is the exponent. An exponential function* is of the form $f(x) = a^x$. It is defined for rational $x = m/n$ by $a^{m/n} = \sqrt[n]{a \cdots a}$ ($m$ factors), $a^{-x} = 1/a^x$, $a^0 = 1$; but this is harder to define for irrational exponents like $a^{\sqrt{2}}$. We have addition and multiplication formulas: $a^{x_1}a^{x_2} = a^{x_1+x_2}$ and $(a^x)^p = a^{px}$.

Given the exponential function $f(x) = a^x$, the logarithm function is the inverse $f^{-1}(x) = \log_a(x)$, as defined in §6.1.†

That is, the equation $y = a^x$ is by definition solved as $x = \log_a(y)$, and we have $\log_a(a^x) = x$ and $a^{\log_a(y)} = y$. Every fact about the exponential function corresponds to an inverse fact about the logarithm. Setting $y_1 = a^{x_1}$, $x_1 = \log(y_1)$; and $y_2 = a^{x_2}$, $x_2 = \log(y_2)$; the addition formula becomes:

$$a^{x_1}a^{x_2} = a^{x_1+x_2} \implies y_1y_2 = a^{\log(y_1)+\log(y_2)}$$

$$\implies \log(y_1y_2) = \log(y_1) + \log(y_2).$$

Setting $y = a^x$, $x = \log(y)$, the multiplication formula becomes:

$$(a^x)^p = a^{px} \implies y^p = a^{p\log(y)}$$

$$\implies \log(y^p) = p \log(y).$$

**EXAMPLE**: Expand the expression $\log \sqrt[4]{x^2+4}$ as much as possible into a sum of simple terms. Using the addition and multiplication formulas:

$$\log \sqrt[4]{x^2+4} = \log ((x+1)(x-1))^{1/2}$$

$$= \frac{1}{2} (\log(x+1) + (-1) \log(x-1))$$

$$= \frac{1}{2} \log(x+1) - \frac{1}{2} \log(x-1).$$

Notes by Peter Magyar magyar@math.msu.edu

*Do not confuse this with a power function of the form $f(x) = x^p$.

†If the base $a$ is understood, we write simply $\log y$. In science and engineering literature, if there is no base specified, we assume the base $a = 10$. 

**Natural exponential and logarithm.** In the physical world, an exponential function \( f(t) = a^t \) typically appears as the size of a population which is self-reproducing. This means the population growth rate, the number of births per unit time, is proportional to the current population size:

\[ f'(t) = c f(t). \]

It is a fact (proven below) that any exponential function \( f(t) = a^t \) satisfies this equation for some constant \( c. \)

The *natural exponential function* uses the unique choice of base \( a = e = 2.718 \ldots \) which makes the above constant \( c = 1 \). That is, if we write \( f(x) = \exp(x) = e^x \), then:

\[ f'(x) = f(x), \quad \exp'(x) = \exp(x), \quad (e^x)' = e^x. \]

The *natural logarithm* is the inverse function of \( f(x) = \exp(x) \), namely \( f^{-1}(x) = \ln(x) = \log_e(x) \), so \( y = e^x \) means \( x = \ln(y) \). As in §6.1, to find the derivative of \( \ln(x) \), we differentiate \( x = \exp(\ln(x)) \):

\[ 1 = \exp'(\ln(x)) \cdot \ln'(x) = \exp(\ln(x)) \ln'(x) = x \ln'(x) \implies \ln'(x) = \frac{1}{x}. \]

Amazingly, though the definition of \( \ln(x) \) was complicated, its derivative is the extremely simple function \( \frac{1}{x} \).

**example:** Find the derivative of \( f(x) = \ln(x^2+1) \). Taking outside function \( \ln(x) \) with \( \ln'(x) = \frac{1}{x} \), and inside function \( x^2+1 \), the Chain Rule gives:

\[ f'(x) = \ln'(x^2+1) \cdot (x^2+1)' = \frac{1}{x^2+1} \cdot (2x) = \frac{2x}{x^2+1}. \]

**example:** Find the derivative of \( f(x) = \ln(\sin(x)) \). From the Chain Rule;

\[ f'(x) = \ln'(\sin(x)) \cdot \sin'(x) = \frac{1}{\sin(x)} \cdot \cos(x) = \cot(x). \]

**example:** Find the derivative of \( f(x) = \frac{(x+1)^3 \sin^2(x)}{(2x+1)^5} \) using the shortcut of *logarithmic differentiation*. Take log of both sides, turning products into sums, then differentiate:

\[
\begin{align*}
\ln f(x) &= 3 \ln(x+1) + 2 \ln(\sin(x)) - 5 \ln(2x+1) \\
(\ln f(x))' &= 3 \frac{1}{x+1} + 2 \frac{1}{\sin(x)} \cos(x) - 5 \frac{1}{2x+1}(2) \\
\frac{1}{f(x)} f'(x) &= 3 \frac{1}{x+1} + 2 \frac{1}{\sin(x)} \cos(x) - 5 \frac{1}{2x+1}(2) \\
f'(x) &= (x+1)^3 \sin^2(x) \left( \frac{3}{x+1} + 2 \cot(x) - \frac{10}{2x+1} \right).
\end{align*}
\]

---

\( ^1 \)Mathematical laws in science are typically stated in such *differential equations*, in which an unknown function \( f(t) \) has a specified relation with its rate of change \( f'(t) \), its acceleration \( f''(t) \), etc. For example, Newton’s law of universal gravitation is essentially \( f''(t) = -c/f(t)^2 \).
Logarithms and integrals. Reversing our new basic derivative $\ln'(x) = \frac{1}{x}$, we see
\[ \int \frac{1}{x} \, dx = \ln(x) + C, \] the antiderivative of $\frac{1}{x} = x^{-1}$, a key function which we previously
could not integrate. (The usual power function formula would give the nonsense answer
\[ \int x^{-1} \, dx = \frac{1}{1-1} x^{1+1} = 1 x^0. \] Thus the Second Fundamental Theorem of Calculus
(§4.3) tells us, for $a, b > 0$:
\[ \int_a^b \frac{1}{x} \, dx = [\ln(x)]_{x=a}^{x=b} = \ln(b) - \ln(a). \]
To extend to negative $x$, we use:
\[ \int \frac{1}{x} \, dx = \ln |x| + C \quad \Rightarrow \quad \int_a^b \frac{1}{x} \, dx = \ln|b| - \ln|a| \quad \text{for } a, b < 0. \]
Geometrically, $\ln(x) = \int_1^x \frac{1}{t} \, dt$ is the area under $y = \frac{1}{t}$ and above the interval $[1, x]$.

Calculators and computers need an approximation algorithm to compute values of $\ln(x)$
more efficiently than just guessing solutions of $e^y = x$. The integral above allows us to
approximate a natural logarithm as a Riemann sum. For example, to compute
\[ \ln(2) = 0.693147 \ldots, \]
split the interval $[1, 2]$ into $n = 100$ increments of size $\Delta x = \frac{2-1}{n} = 0.01$, take sample
points $x_i = 1 + i\Delta x$ for $i = 1, \ldots, n$, and compute the right Riemann sum:
\[ \ln(2) \approx \frac{1}{1.01}(0.01) + \frac{1}{1.02}(0.01) + \cdots + \frac{1}{1.99}(0.01) + \frac{1}{2.00}(0.01) \approx 0.691, \]
which is accurate to two decimal places. The Midpoint Method, which samples midpoints
$x_i = 1 + i\Delta x - \frac{\Delta x}{2}$, gives five decimal places without more computation:
\[ \ln(2) \approx \frac{1}{1.005}(0.01) + \frac{1}{1.015}(0.01) + \cdots + \frac{1}{1.985}(0.01) + \frac{1}{1.995}(0.01) \approx 0.693144. \]

EXAMPLE: Compute $\int_a^b \frac{\ln(x)}{x} \, dx$. Although there does not appear to be any outside or
inside function, we see that $\frac{\ln(x)}{x} = \frac{\ln(x)}{x} \cdot \frac{1}{x} = \ln(x) \cdot \ln'(x)$, so we can use the substitution
$u = \ln(x)$, $du = \frac{1}{x} \, dx$:
\[ \int \ln(x) \frac{1}{x} \, dx = \int u \, du = \frac{1}{2} u^2 = \frac{1}{2} \ln^2(x). \]
Thus, $\int_a^b \frac{\ln(x)}{x} \, dx = \frac{1}{2} \ln^2(b) - \frac{1}{2} \ln^2(a)$. 

\[ \begin{array}{c}
\text{y} = 1/x \\
A = \ln(x)
\end{array} \]
EXAMPLE: A tricky integral: \( \int \sec(x) \, dx \). There seems to be no convenient substitution, but an amazing trick introduces \( \sec^2(x) = \tan'(x) \) and \( \sec(x) \tan(x) = \sec'(x) \):

\[
\int \sec(x) \, dx = \int \sec(x) \frac{\sec(x) + \tan(x)}{\sec(x) + \tan(x)} \, dx = \int \frac{1}{\sec(x) + \tan(x)} \left( \sec^2(x) + \sec(x) \tan(x) \right) \, dx
\]

This is perfect to substitute \( u = \sec(x) + \tan(x) \), \( du = (\sec^2(x) + \sec(x) \tan(x)) \, dx \):

\[
\int \frac{1}{u} \, du = \ln|u| = \ln|\sec(x) + \tan(x)|.
\]

EXAMPLE: Time flies.\(^5\) Suppose that the subjective length of each day is equal to the fraction of your past life it represents. Thus, when you are 1 year old, an extra day feels like \( \frac{1}{365} \) of a year; but when you are 2 years old, it only feels like \( \frac{1}{365} \) of a year; and after \( t \) years, it feels like \( \frac{1}{365} \). Adding up the subjective lengths of all the days from \( t = 1 \) to \( x \) years, denoting \( \Delta t = \frac{1}{365} \):

\[
s(x) \approx \frac{1}{2} \Delta t + \frac{1}{1+\Delta t} \Delta t + \frac{1}{1+2\Delta t} \Delta t + \frac{1}{1+3\Delta t} \Delta t + \cdots + \frac{1}{x} \Delta t.
\]

But this is just a Riemann sum for \( s(x) = \int_1^x \frac{1}{t} \, dt = \log(x) \). This means your subjective years increase logarithmically, and the time from any age 1 to age \( e \approx 2.7 \) feels the same as from age 10 to age 10e \( \approx 27 \). Is it true?

Proofs. To formally prove the basic facts about exponentials and logarithms, we start from the one connection between these complicated functions and an elementary function: \( \ln'(x) = \frac{1}{x} \).

We now forget everything we previously stated about exponentials and logarithms, and build up our definitions from scratch, proving all properties.

**Definition.** For a given \( x > 0 \), we let: \( \ln(x) = \int_1^x \frac{1}{t} \, dt \).

That is, having forgotten our previous definition of \( \ln(x) \), we take the symbol \( \ln(x) \) to mean the given integral, which we can compute to arbitrary accuracy with Riemann sums. Given this, the First Fundamental Theorem (§4.3) immediately proves \( \ln'(x) = \frac{1}{x} \). Next:

**Theorem:** (a) \( \ln(x_1 x_2) = \ln(x_1) + \ln(x_2) \); (b) \( \ln(x^p) = p \ln(x) \).

**Proof.** (a) For a constant \( k > 0 \), the derivative of \( \ln(kx) \) is:

\[
\ln(kx)' = \ln'(kx) \cdot k = \frac{1}{kx} \cdot k = \frac{1}{x} = \ln'(x).
\]

Since \( \ln(kx) \) and \( \ln(x) \) are both antiderivatives of \( \frac{1}{x} \), we must have \( \ln(kx) = \ln(x) + C \) for some constant \( C \) (§3.9 Antiderivative Theorem). Setting \( x = 1 \), we get \( \ln(k) = \ln(1) + C = C \), i.e. \( C = \ln(k) \). Thus, \( \ln(kx) = \ln(x) + \ln(k) = \ln(k) + \ln(x) \), which becomes the desired formula if we let \( k = x_1 \) and \( x = x_2 \).

(b) We use the same steps, starting from \( \ln(x^p)' = \frac{1}{x^p} p x^{p-1} = \frac{p}{x} = (p \ln(x))' \).

**Definition:** The function \( f(x) = \ln(x) \) is one-to-one for \( x > 0 \), so it has an inverse function \( f^{-1}(x) = \ln^{-1}(x) \). We name this inverse \( \exp(x) = \ln^{-1}(x) \).

Indeed, since \( \ln(x) = \frac{1}{x} > 0 \) for all \( x > 0 \), we know that \( \ln(x) \) is increasing, i.e. \( x_1 < x_2 \) guarantees \( \ln(x_1) < \ln(x_2) \); thus \( \ln(x) \) is necessarily one-to-one. The Inverse Theorem (§6.1) immediately proves \( \exp(\ln(x)) = x \) and \( \ln(\exp(x)) = x \).

\( ^5 \)Time flies like an arrow. Fruit flies like a banana.
Theorem: \( \exp'(x) = \exp(x) \)

Proof. As in the Inverse Derivative Theorem (§6.1), differentiating \( x = \ln(\exp(x)) \) gives:

\[
1 = [\ln(\exp(x))]' = \ln'(\exp(x)) \cdot \exp'(x) = \frac{1}{\exp(x)} \exp'(x) \quad \implies \quad \exp'(x) = \exp(x).
\]

The exponential addition and multiplication formulas, \( \exp(x_1) \exp(x_2) = \exp(x_1 + x_2) \) and \( \exp(x)^p = \exp(px) \), follow from reversing our previous reasoning which proved the logarithm formulas from the exponential ones. We define the number \( e = \exp(1) \), i.e. the unique number such that: \( \int_1^e \frac{1}{t} \, dt = 1 \). Finally, we define a general exponential function as \( a^x = \exp(\ln(a)x) \). The Chain Rule then gives \( (a^x)' = \exp(\ln(a)x) \cdot \ln(a) = \ln(a) a^x \).