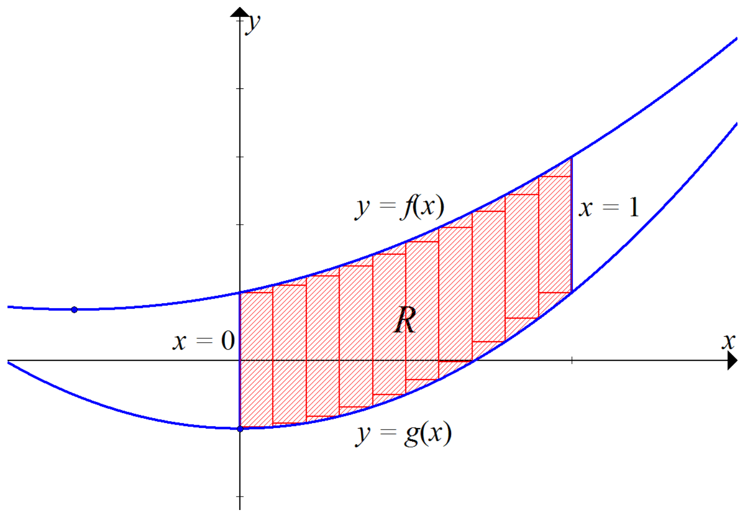
Area Between Curves

Region between two parabolas. We have seen that geometrically, the integral $\int_a^b f(x)\,dx$ computes the area between a curve $y = f(x)$ and an interval $x \in [a, b]$ on the x-axis (with area below the axis counted negatively). In Calculus II, we will show the versatility of the integral to compute all kinds of areas, lengths, volumes: almost any measure of size for a geometric object.

In this section, we compute more general areas: those between two given curves $y = f(x)$ and $y = g(x)$, usually with no boundary on the x-axis.

EXAMPLE: Consider the region with top boundary $y = f(x) = x^2{+}x{+}1$, bottom boundary $y = g(x) = 2x^2{-}1$, left boundary the y-axis $x = 0$, right boundary $x = 1$:*

$$\begin{aligned} R &= \{\,(x, y) \text{ with } g(x) \le y \le f(x) \text{ and } x \in [0, 1]\,\}. \\ &= \{\,(x, y) \text{ with } 2x^2{-}1 \le y \le x^2{+}x{+}1 \text{ and } 0 \le x \le 1\,\}. \end{aligned}$$



Here $y = g(x) = 2x^2{-}1$ is a standard parabola shifted downward, with minimum point $x = 0$. The curve $y = f(x) = x^2{+}x{+}1$ is roughly like its leading term $y = x^2$, a parabola opening upward; its minimum point satisfies $(x^2{+}x{+}1)' = 2x{+}1 = 0$, i.e. $x = -\frac{1}{2}$.

To compute the area of R, we use the same geometric-numerical strategy as for the region under a single curve: split R into n thin vertical slices of width $\Delta x = \frac{1}{n}$, each approximately a rectangle; then add up the rectangle areas and take the limit as n becomes larger and larger. In the interval $x \in [0, 1]$, we take sample points $x_1, \ldots, x_n$, one in each Δx increment. The slice at position x_i has height equal to the ceiling minus the floor, $f(x_i) - g(x_i)$, so:

$$\text{area of slice} \;\approx\; (\text{height})\times(\text{width}) \;=\; (f(x_i) - g(x_i))\,\Delta x,$$

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*We specify the region as the set of all points (x, y) which satisfy the given conditions.

and the total area is:

$$A_{[0,1]} \;=\; \lim_{n\to\infty}\sum_{i=1}^{n}(f(x_i)-g(x_i))\,\Delta x \;=\; \int_0^1 (f(x)-g(x))\,dx$$

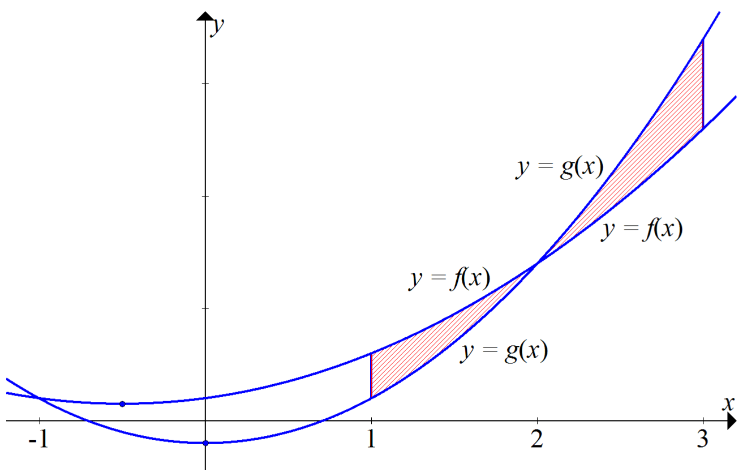
$$=\; \int_0^1 (x^2{+}x{+}1)-(2x^2{-}1)\,dx \;=\; \Big[2x{+}\tfrac{1}{2}x^2{-}\tfrac{1}{3}x^3\Big]_{x=0}^{x=1} \;=\; \tfrac{13}{6}\,.$$

EXAMPLE: Next, consider the region between the same curves $y = f(x) = x^2{+}x{+}1$ and $y = g(x) = 2x^2{-}1$, but above the interval $x \in [1, 3]$. To picture the region without a calculator, we determine the intersection points where the curves cross:

$$f(x) = g(x) \quad\Longleftrightarrow\quad x^2{+}x{+}1 = 2x^2{-}1 \quad\Longleftrightarrow$$

$$x^2{-}x{-}2 = 0 \quad\Longleftrightarrow\quad x = \tfrac{1\pm\sqrt{1^2-4(1)(-2)}}{2(1)} = \tfrac{1\pm 3}{2} = -1 \text{ or } 2$$

by the Quadratic Formula. Only $x = 2$ is relevant for our region above $x \in [1, 3]$. At $x = 1$ we have $g(1) < f(1)$, so to the left of $x = 2$, our region is defined by $g(x) \le y \le f(x)$. At $x = 3$, we have $f(3) < g(3)$, so to the right of $x = 2$, it is $f(x) \le y \le g(x)$:



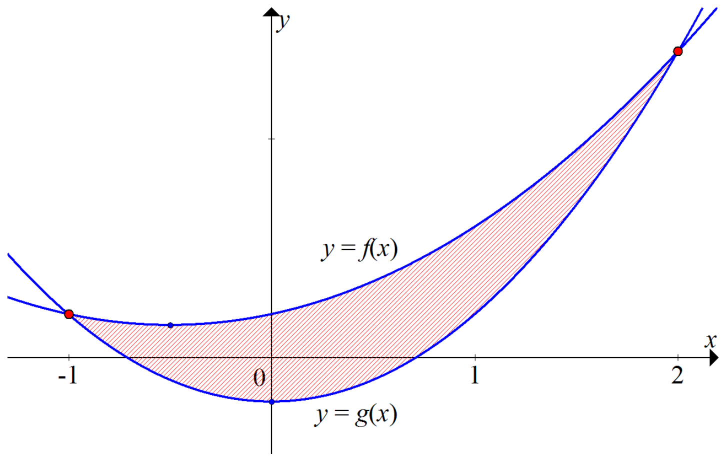
Repeating our previous area formula for the two parts of our region gives:

$$A_{[1,3]} \;=\; A_{[1,2]} + A_{[2,3]} \;=\; \int_1^2 (f(x)-g(x))\,dx + \int_2^3 (g(x)-f(x))\,dx$$

$$=\; \int_1^2 (2{+}x{-}x^2)\,dx \;+\; \int_2^3 (-2{-}x{+}x^2)\,dx \;=\; \tfrac{7}{6} + \tfrac{11}{6} \;=\; 3.$$

EXAMPLE: Finally, we consider the same curves $y = f(x) = x^2+x+1$ and $y = g(x) = 2x^2-1$, but we take the entire finite region between them:

$$R' = \{(x, y) \text{ with } g(x) \leq y \leq f(x)\}.$$



Here the top boundary is $y = x^2 + x + 1$ and the bottom boundary is $y = 2x^2 - 1$, but we have not specified an x interval. However, we have already computed the intersection points $x = -1$ and $x = 2$, and the curves do not enclose any finite regions beyond these points. Thus:

$$A_{[-1,2]} \;=\; \int_{-1}^{2} (f(x) - g(x))\, dx = \tfrac{9}{2}\,.$$

We can generalize the above examples in:

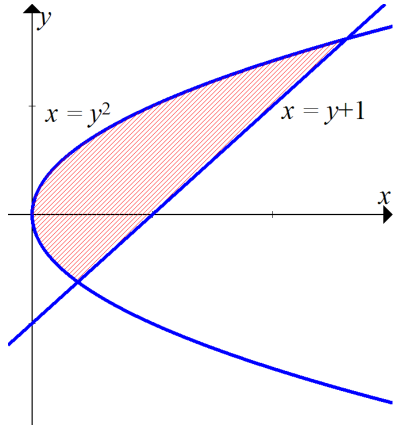
> *Theorem:* The area of the region enclosed between $f(x)$ and $g(x)$ for $x \in [a, b]$ is: $A \;=\; \displaystyle\int_a^b \big|f(x) - g(x)\big|\; dx\,.$

The absolute value signs ensure we take the integral of top minus bottom, regardless of which is which. In practice, we must find the intersection points where $f(x) = g(x)$, which split the integral into intervals where $g(x) \leq f(x)$ versus $f(x) \leq g(x)$.

Integrating with respect to y. Consider the region:

$$R \;=\; \{(x, y) \text{ with } y^2 \leq x \leq y+1\}\,.$$

Here the boudary curves are naturally graphs in which y is the independent variable: the right boundary is the line $x = f(y) = y+1$; and the left boundary is $x = g(y) = y^2$, a parabola opening to the right.



Understand: it is merely by habit that we consider y as a function of x. We can make x a function of y instead if it is more convenient, and the same formulas will work if we switch the roles of x and y. Thus, we find the intersection points: $y{+}1 = y^2$ when $y = \frac{1\pm\sqrt{5}}{2}$ by the Quadratic Formula. The area as:

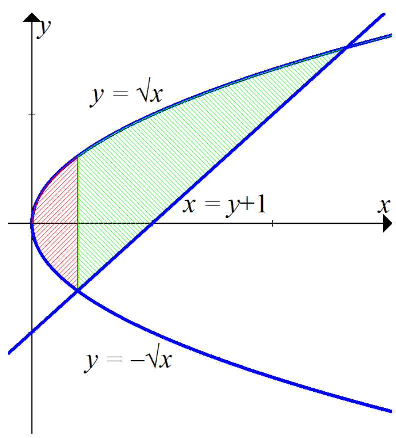
$$A \;=\; \int_{\frac{1-\sqrt{5}}{2}}^{\frac{1+\sqrt{5}}{2}} (y{+}1) - (y^2)\,dy \;=\; \left[\tfrac{1}{2}y^2 + y - \tfrac{1}{3}y^3\right]_{y=\frac{1-\sqrt{5}}{2}}^{y=\frac{1+\sqrt{5}}{2}} \;=\; \tfrac{5}{6}\sqrt{5}.$$

Here $((y{+}1) - (y^2))\,dy$ represents the area of the *horizontal* slice of the region at height y, with thickness dy.

To check this, we re-do it from our usual perspective, using x as the independent variable. This makes it more complicated, since we must consider the region as having three boundary graphs: upper boundary $y = \sqrt{x}$, lower right boundary $y = x{-}1$, and lower left boundary $y = -\sqrt{x}$. The intersection points are:

- Between $y = \sqrt{x}$ and $y = x{-}1$: $x = \frac{3+\sqrt{5}}{2}$ (upper right corner)
- Between $y = -\sqrt{x}$ and $y = x{-}1$: $x = \frac{3-\sqrt{5}}{2}$ (lower middle corner)
- Between $y = \sqrt{x}$ and $y = -\sqrt{x}$: $x = 0$ (left end)

These split the region into left and right parts:



The area is:

$$A \;=\; \int_0^{\frac{3-\sqrt{5}}{2}} (\sqrt{x}) - (-\sqrt{x})\,dx + \int_{\frac{3-\sqrt{5}}{2}}^{\frac{3+\sqrt{5}}{2}} (\sqrt{x}) - (x{-}1)\,dx\,,$$

which after much algebra gives the same answer as before: $\frac{5}{6}\sqrt{5}$.