Math 133 # Method for Convergence Testing **Stewart §11.7**

For a series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$, determine if it converges toward a limit as we add more terms, or diverges (either to $\pm\infty$ or oscillating).

0. If $\lim_{n\to\infty} a_n \neq 0$, then the series diverges by the n-th Term Test (Vanishing Test).

1. Try to manipulate the series into a Standard Series:

   - Geometric series: $\sum_{n=1}^{\infty} cr^{n-1} = c+cr+cr^2+cr^3+\cdots = \begin{cases} \frac{c}{1-r} & \text{for } |r|<1 \\ \text{diverges} & \text{for } |r| \geq 1. \end{cases}$
   - Standard p-series: $\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots = \begin{cases} \text{converges} & \text{for } p>1 \\ \text{diverges} & \text{for } p \leq 1. \end{cases}$

2. If a_n is a fraction, estimate with a simpler fraction b_n, often a standard series, by taking only the *largest* term from the numerator and denominator of a_n.
   Convergence of $\sum a_n$ is usually same as convergence of $\sum b_n$. Justify with a Test:

   - Direct Comparison Test (positive a_n)
     - Ceiling $0 \leq a_n \leq c_n$ where $\sum c_n$ converges $\implies$ $\sum a_n$ also converges.
     - Floor $0 \leq d_n \leq a_n$ where $\sum d_n$ diverges $\implies$ $\sum a_n$ also diverges.

     The ceiling c_n or floor d_n will usually be closely related to the estimate b_n.
   - Limit Comparison Test: Determine $L = \lim_{n\to\infty} a_n / b_n$.
     - $|L| < \infty$ and $\sum b_n$ converges $\implies$ $\sum a_n$ also converges.
     - $|L| > 0$ and $\sum b_n$ diverges $\implies$ $\sum a_n$ also diverges.
       [Compare with $(L-\epsilon)b_n < a_n < (L+\epsilon)b_n$ for $n > N$].*

3. Try the Integral Test if a_n is positive and fairly simple, but not comparable to a standard series: e.g. $\frac{1}{n\ln(n)}$. For a positive decreasing function $f(x)$ with $a_n = f(n)$, compute improper integral $\int_1^{\infty} f(x)\,dx = \lim_{N\to\infty} \int_1^N f(x)\,dx = \lim_{N\to\infty} F(N) - F(1)$.

   - $\int_1^{\infty} f(x)\,dx$ converges $\implies$ $\sum a_n$ also converges $[\sum_{n=1}^{\infty} a_n \leq a_1 + \int_1^{\infty} f(x)\,dx]$.
   - $\int_1^{\infty} f(x)\,dx$ diverges $\implies$ $\sum a_n$ also diverges $[\sum_{n=1}^{\infty} a_n \geq \int_1^{\infty} f(x)\,dx]$.

4. Try the Ratio Test if a_n has a growing number of factors, for example if it contains r^n or $n!$. Determine $\lim_{n\to\infty} |a_{n+1}/a_n| = L$.

   - $L < 1$ $\implies$ $\sum a_n$ converges $[\,|a_n| \leq c(L+\epsilon)^n$ for $n > N]$.
   - $L > 1$ $\implies$ $\sum a_n$ diverges $[\,|a_n| \geq c(L-\epsilon)^n$ for $n > N]$.
   - $L = 1$ $\implies$ no conclusion. [e.g. any standard p-series]

5. If $\sum a_n$ has positive and negative terms, try:

   - Absolute Convergence: $\sum |a_n|$ converges $\implies$ $\sum a_n$ also converges.
   - Alternating Series: $a_n = (-1)^{n-1} b_n$ with $b_n \geq 0$:
     $\lim_{n\to\infty} b_n = 0$, b_n decreasing $\implies$ $\sum a_n$ converges.
     Error estimate: $\sum_{n=1}^{2N} a_n \leq \sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{2N} a_n + b_{2N+1}$.

Notes by Peter Magyar `magyar@math.msu.edu`

*Most later tests are proved by reducing to a Direct Comparison, specified in [brackets]