We have one more important test for convergence of an infinite series $\sum_{n=1}^{\infty} a_n$. This test does not require us to choose a comparison series: instead, we test the ratio of each term a_n compared to the next term a_{n+1}.

Ratio Test: Suppose $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$.

- If $L < 1$, then $\sum_{n=1}^{\infty} a_n$ converges.
- If $L > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.
- If $L = 1$, then this test fails to determine convergence.

Proof: Assuming $a_n > 0$, the limit $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$ means that, for any small number $\epsilon > 0$, we can take a starting point N so that for all $n \geq N$, we have:

$$\begin{aligned} L-\epsilon \quad &\leq \quad \frac{a_{n+1}}{a_n} \quad \leq \quad L+\epsilon \\ a_n(L-\epsilon) \quad &\leq \quad a_{n+1} \quad \leq \quad a_n(L+\epsilon)\,. \end{aligned}$$

Iterating this inequality gives: $c_1(L-\epsilon)^n \leq\ a_n \leq\ c_2(L+\epsilon)^n$ for some constants c_1, c_2.*

If $L < 1$, we take ϵ small enough that $L+\epsilon < 1$, and we compare $\sum a_n$ to the convergent ceiling series $\sum c_2(L+\epsilon)^n$. If $L > 1$, we take ϵ small enough that $L-\epsilon > 1$, and we compare $\sum a_n$ to the divergent floor series $\sum c_2(L-\epsilon)^n$. If $L = 1$, adding any ϵ produces a divergent ceiling, and subtracting any ϵ produces a convergent floor, neither of which would constrain the original series. Finally, for the general case where the a_n's may be positive or negative, the above argument shows $\sum |a_n|$ converges, which implies $\sum a_n$ converges by §11.6 Part II. Q.E.D.

The Ratio Test is most useful when a_n is a product of a growing number of factors, which will mostly cancel out in $\frac{a_{n+1}}{a_n}$.

EXAMPLE: Determine the convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2^n}$.

We did this one in §11.4 by finding a tricky comparison series. The Ratio Test naturally applies here, because $a_n = \frac{n^2}{2^n} = (n)(n)(\frac{1}{2})\cdots(\frac{1}{2})$ has more and more factors as n gets larger. We have:

$$L \;=\; \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \;=\; \lim_{n\to\infty} \frac{(n+1)^2}{2^{n+1}} \Big/ \frac{n^2}{2^n} \;=\; \lim_{n\to\infty} \frac{(n+1)^2}{n^2} \Big/ \frac{2^{n+1}}{2^n} \;=\; \frac{1}{2}\,.$$

Since $L = \frac{1}{2} < 1$, the Test shows $\sum a_n$ converges.

EXAMPLE: Determine the convergence of $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{x^{2n}}{n!}$, where x is a given number and we use the factorial notation $n! = (n)(n-1)(n-2)\cdots(2)(1)$. Again, the terms have a large number of factors, so we use the Ratio Test:

$$L \;=\; \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \;=\; \lim_{n\to\infty} \frac{x^{2(n+1)}}{(n+1)!} \Big/ \frac{x^{2n}}{n!} \;=\; \lim_{n\to\infty} \frac{x^2}{n+1} \;=\; 0\,.$$

Since $L = 0 < 1$, the Test shows $\sum a_n$ converges.

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*Specifically: $a_n \leq a_{n-1}(L+\epsilon) \leq a_{n-2}(L+\epsilon)^2 \leq \cdots \leq a_N(L+\epsilon)^{n-N} = \frac{a_N}{(L+\epsilon)^N}(L+\epsilon)^n$.