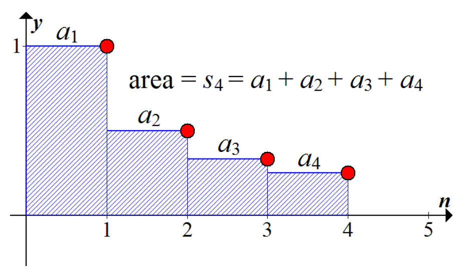
# Series

**Sequences and series.** To any sequence $\{a_n\}_{n=1}^{\infty}$ we associate another sequence $\{s_n\}_{n=1}^{\infty}$, called the *series* of sums of $\{a_n\}$, defined by:

$$s_n \;=\; a_1 + a_2 + \cdots + a_n \;=\; \sum_{i=1}^{n} a_i\,.$$

That is, $s_1 = a_1$, $s_2 = a_1 + a_2$, $s_3 = a_1 + a_2 + a_3$, and in general $s_n$ is the sum of the first $n$ entries of $\{a_n\}$. The sigma notation $\sum_{i=1}^{n} a_i$ is a convenient shorthand for taking each integer value $i = 1, 2, \ldots, n$, substituting it into the expression $a_i$, and adding all the resulting quantities (see §4.1 Pt 2). We can change the index letters arbitrarily: $s_N = \sum_{n=1}^{N} a_n$ defines the same series as $s_n = \sum_{i=1}^{n} a_i$.

EXAMPLE: If $a_n = \frac{1}{n}$, then $s_1 = 1$, $s_2 = 1 + \frac{1}{2} = \frac{3}{2}$, $s_3 = 1 + \frac{1}{2} + \frac{1}{3} = \frac{11}{6}$, etc. The general term $s_n = \sum_{i=1}^{n} \frac{1}{i} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ has no elementary formula, which is typical for series, even when $a_n$ is quite simple. Geometrically, $s_n$ is the area under the bar graph of $\{a_n\}$ above the interval $[0, n]$.




We can also take the infinite sum, which is defined as a limit of finite sums:

$$\sum_{i=1}^{\infty} a_i \;=\; \lim_{n\to\infty} \sum_{i=1}^{n} a_i \;=\; \lim_{n\to\infty} s_n\,.$$

This limit, converging or diverging, is the total area under the bar graph of $\{a_n\}_{n=1}^{\infty}$.

EXAMPLE: The general purpose of a series is to express a complicated quantity as an infinite sum of simple quantities, $s = \sum_{i=1}^{\infty} a_i$, so that the finite sums $s_n = \sum_{i=1}^{n} a_i$ are approximations. A familiar example of this is decimal notation: an irrational real number is a complicated quantity with an infinite amount of detail in its digits, which are equivalent to the sum of a certain series.

Given a sequence of digits $\{d_n\}_{n=1}^{\infty}$ with $d_n \in \{0, 1, \ldots, 9\}$, we have the number:

$$s \;=\; 0.d_1 d_2 d_3 \cdots \;=\; \frac{d_1}{10^1} + \frac{d_2}{10^2} + \frac{d_3}{10^3} + \cdots \;=\; \sum_{n=1}^{\infty} \frac{d_n}{10^n}\,.$$

By definition, an infinite decimal is the limit of its finite decimal approximations, the number approached as we add more digits. A trivial example is the repeating decimal $s = 0.999\cdots$, which clearly gets as close as desired to 1 as we take more

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digits, so $s = 1$. For a more complicated pattern of digits, the series converges to some complicated real number (see §11.4).

EXAMPLE: We will eventually (§11.9, 11.10) develop powerful methods to write familiar numbers and functions as infinite series. Two outstanding formulas are:

$$\pi \;=\; 4(1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \cdots), \qquad \sin(x) \;=\; x - \tfrac{1}{3!}x^3 + \tfrac{1}{5!}x^5 - \tfrac{1}{7!}x^7 + \cdots.$$

It is formulas like these that allow machines to compute complicated transcendental quantities using only the four arithmetic operations (which are all that can easily be built into a logic circuit).

**Geometric progression on a chessboard.** A classic puzzle says: if we put one kernel of wheat on the first square of a chessboard, then two kernels on the second square, then four on the third square, and we keep doubling until the 64th square, how many kernels on the whole board?

We start with the sequence $a_n$ = number of kernels on the $n$th square, defined recursively by $a_1 = 1,\ a_n = 2a_{n-1}$, which leads to the explicit formula $a_n = 2^{n-1}$. This is called an exponential sequence or *geometric sequence*.* The associated *geometric series* is $s_n$ = total number of kernels on the first $n$ squares:

$$s_n \;=\; \sum_{i=1}^{n} 2^{i-1} \;=\; 1 + 2^1 + 2^2 + \cdots + 2^{n-2} + 2^{n-1}.$$

Surprisingly, we can find a simple formula for $s_n$ as follows:

$$\begin{array}{rcccccccccc} 2s_n & = & & & 2^1 & + & 2^2 & + \cdots + & 2^{n-1} & + & 2^n \\ -s_n & = & -1 & - & 2^1 & - & 2^2 & - \cdots - & 2^{n-1} & & \end{array}$$

Adding these, the two sides become:

$$(2-1)s_n \;=\; 2a_n - a_1 \;=\; 2^n - 1 \qquad \Longrightarrow \qquad s_n \;=\; \frac{2^n - 1}{2 - 1} \;=\; 2^n - 1\,.$$

Therefore, the answer to our puzzle is $s_{64} = 2^{64} - 1$ kernels, which is enough wheat to fill the bowl of Spartan Stadium.

Now we change the problem: we put one ounce of gold on the first square, half an ounce on the second square, a quarter ounce on the third square, and so on until the 64th. What is the total weight of gold on the board?

The weight on the $n$th square is given by another geometric (i.e. exponential) sequence $a_n = \left(\frac{1}{2}\right)^{n-1}$, and the total weight on the first $n$ squares is the geometric series $s_n = \sum_{i=1}^{n} \left(\frac{1}{2}\right)^{i-1}$. We use the same trick as before:

$$\begin{array}{rcccccccccc} s_n & = & 1 & + & \frac{1}{2} & + & \left(\frac{1}{2}\right)^2 & + \cdots + & \left(\frac{1}{2}\right)^{n-1} & & \\ -\frac{1}{2}s_n & = & & - & \frac{1}{2} & - & \left(\frac{1}{2}\right)^2 & - \cdots - & \left(\frac{1}{2}\right)^{n-1} & - & \left(\frac{1}{2}\right)^n \end{array}$$

$$(1 - \tfrac{1}{2})s_n \;=\; a_1 - \tfrac{1}{2}a_n \;=\; 1 - \left(\tfrac{1}{2}\right)^n \qquad \Longrightarrow \qquad s_n \;=\; \frac{1 - \left(\frac{1}{2}\right)^n}{1 - \frac{1}{2}}\,.$$

*Geometric progression: $2$ = length of a segment, $2^2$ = area of a square, $2^3$ = volume of cube.

Thus, the total weight is $s_{64} = \frac{1-(\frac{1}{2})^{64}}{1-\frac{1}{2}}$. Since $\left(\frac{1}{2}\right)^{64}$ is a tiny, negligeable quantity, this is very close to $\frac{1}{1-\frac{1}{2}} = 2$, meaning the first square has just about the same total weight as the other 63 squares. In fact, adding more squares would barely change the total, since the limit is:

$$\sum_{i=1}^{\infty} \left(\tfrac{1}{2}\right)^{i-1} \;=\; \lim_{n\to\infty} s_n \;=\; \lim_{n\to\infty} \frac{1-\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}} \;=\; \frac{1-0}{1-\frac{1}{2}} \;=\; 2\,.$$

**Geometric sequences and series.** A general geometric sequence starts with an initial value $a_1 = c$, and subsequent terms are multiplied by the ratio $r$, so that $a_n = ra_{n-1}$; explicitly, $a_n = cr^{n-1}$. The same trick as above gives a formula for the corresponding geometric series. We have $s_n - rs_n = c - cr^n$, so:

$$\boxed{s_n \;=\; \sum_{i=1}^{n} cr^{i-1} \;=\; c + cr + cr^2 + \cdots + cr^{n-1} \;=\; c\,\frac{1-r^n}{1-r}\,.}$$

(Notice that the power $r^n$ is one larger than in the last term $cr^{n-1}$.) This ingenious formula is known as the *sum of a finite geometric series*; the limit is the *sum of an infinite geometric series*:

$$\boxed{\lim_{n\to\infty} s_n \;=\; \sum_{i=1}^{\infty} cr^{i-1} \;=\; c\,\frac{1}{1-r}\,, \quad \text{provided } |r| < 1.}$$

Of couse, the infinite series diverges if $|r| \geq 1$. These formulas are needed again and again in practical problems, especially those involving financal interest rates.

**Manipulating sigma notation.** The geometric series formulas allow us to evaluate any series whose terms involve only exponential functions like $2^n$ or $2^{-n}$, but not power functions like $n$ or $n^2$. To evaluate, we rearrange and manipulate the terms into the form of the geometric sequences which we know.

EXAMPLE: Evaluate the finite sum $\displaystyle\sum_{i=1}^{n} \frac{2^i-1}{3^{i+1}}$, and the infinite sum $\displaystyle\sum_{i=1}^{\infty} \frac{2^i-1}{3^{i+1}}$.

First, we work this out in dot-dot-dot notation:

$$\frac{2^1-1}{3^{1+1}} + \cdots + \frac{2^n-1}{3^{n+1}} \;=\; \left(\frac{2^1}{3^{1+1}} + \cdots + \frac{2^n}{3^{n+1}}\right) - \left(\frac{1}{3^{1+1}} + \cdots + \frac{1}{3^{n+1}}\right)$$

$$= \left(\frac{2}{3^2}\frac{2^0}{3^0} + \cdots + \frac{2}{3^2}\frac{2^{n-1}}{3^{n-1}}\right) - \left(\frac{1}{3^2}\frac{1}{3^0} + \cdots + \frac{1}{3^2}\frac{1}{3^{n-1}}\right)$$

$$= \left(\frac{2}{3^2}\left(\frac{2}{3}\right)^0 + \cdots + \frac{2}{3^2}\left(\frac{2}{3}\right)^{n-1}\right) - \left(\frac{1}{3^2}\left(\frac{1}{3}\right)^0 + \cdots + \frac{1}{3^2}\left(\frac{1}{3}\right)^{n-1}\right)$$

$$= \tfrac{2}{3^2}\,\frac{1-\left(\frac{2}{3}\right)^n}{1-\frac{2}{3}} \;-\; \tfrac{1}{3^2}\,\frac{1-\left(\frac{1}{3}\right)^n}{1-\frac{1}{3}}$$

Here we factored out $\frac{2}{3^2}$ and $\frac{1}{3^2}$ so the remaining factor would be $r^{i-1}$ for some $r$ and $i = 1, \ldots, n$, which we can then evaluate using the geometric series formula.

This computation can be written more compactly in sigma notation:

$$\begin{aligned}
\sum_{i=1}^{n} \tfrac{2^i - 1}{3^{i+1}} &= \sum_{i=1}^{n} \left( \tfrac{2^i}{3^{i+1}} - \tfrac{1}{3^{i+1}} \right) \\
&= \sum_{i=1}^{n} \tfrac{2^i}{3^{i+1}} \; - \; \sum_{i=1}^{n} \tfrac{1}{3^{i+1}} \\
&= \sum_{i=1}^{n} \tfrac{2}{3^2} \tfrac{2^{i-1}}{3^{i-1}} \; - \; \sum_{i=1}^{n} \tfrac{1}{3^2} \tfrac{1}{3^{i-1}} \\
&= \sum_{i=1}^{n} \tfrac{2}{3^2} \left(\tfrac{2}{3}\right)^{i-1} \; - \; \sum_{i=1}^{n} \tfrac{1}{3^2} \left(\tfrac{1}{3}\right)^{i-1} \\
&= \tfrac{2}{3^2} \, \frac{1 - \left(\frac{2}{3}\right)^n}{1 - \frac{2}{3}} \; - \; \tfrac{1}{3^2} \, \frac{1 - \left(\frac{1}{3}\right)^n}{1 - \frac{1}{3}} \, .
\end{aligned}$$

To get the infinite sum, we just remove the terms $(\frac{2}{3})^n$ and $(\frac{1}{3})^n$, since these go to zero as $n \to \infty$.

**Repeating decimals.** An important application of geometric series is to write repeating infinite decimals as fractions. For example, consider:

$$s \;=\; 0.0626262\cdots \;=\; 0.0\overline{62}$$

We can apply the ratio-multiplication trick to cancel the infinite tail of digits:

$$s \;=\; 0.0626262\cdots \,, \qquad \tfrac{1}{100} s \;=\; 0.0006262\cdots$$

$$(1 - \tfrac{1}{100}) s \;=\; 0.062 \qquad \Longrightarrow \qquad \tfrac{99}{100} s \;=\; \tfrac{62}{1000} \qquad \Longrightarrow \qquad s \;=\; \tfrac{62}{990} \, .$$

It is no coincidence that we can use the same trick as before: in fact, this infinite decimal is the sum of a geometric series:

$$\begin{aligned}
s \;&=\; \tfrac{6}{10^2} + \tfrac{2}{10^3} + \tfrac{6}{10^4} + \tfrac{2}{10^5} + \tfrac{6}{10^6} + \tfrac{2}{10^7} + \cdots \;=\; \tfrac{62}{10^3} \;+\; \tfrac{62}{10^5} \;+\; \tfrac{62}{10^7} \;+\; \cdots \\
&= \sum_{i=1}^{\infty} \frac{62}{10^{2i+1}} \;=\; \sum_{i=1}^{\infty} \frac{62}{10^1} \frac{1}{(10^2)^i} \;=\; \sum_{i=1}^{\infty} \frac{62}{10^3} \left(\frac{1}{100}\right)^{i-1} \\
&= \frac{62}{10^3} \frac{1}{1 - \frac{1}{100}} \;=\; \frac{62}{1000} \frac{100}{99} \;=\; \frac{62}{990} .
\end{aligned}$$

Another example: $1.5626262\cdots \;=\; 1.5 + 0.0626262\cdots \;=\; \frac{15}{10} + \frac{62}{990} \;=\; \frac{1547}{990}$.

We can carry out such reasoning for any infinite decimal which starts with arbitrary digits, then becomes repeating. Conversely, for any fraction $a/b$, the long division $b\overline{\smash{)}a}$ will produce a repeated remainder after at most $b{-}1$ steps, leading to a repeating decimal with a period of at most $b{-}1$. E.g. $1/7 = 0.\overline{142857}$ has remainders $3, 2, 6, 4, 5, 1, 3, 2, \ldots$.

Thus, any infinite decimal represents a real number, and the repeating decimals represent precisely the rational numbers (fractions)! For example, since we know $\sqrt{2} = 1.4142135623730950488\cdots$ is an irrational number, not equal to any fraction, its decimal digits will never settle into an infinitely repeating pattern.

**Geometric power series.** We can take the ratio in a geometric series to be a variable, obtaining a function called a *power series*:

$$g(x) \;=\; \sum_{n=0}^{\infty} x^n \;=\; 1 + x + x^2 + x^3 + \cdots$$

(Traditionally, the index starts at $n = 0$, so the first term is $x^0 = 1$.) By definition, this means that for the input $x = r$, the output is:

$$g(r) \;=\; \sum_{n=0}^{\infty} r^n \;=\; \lim_{N\to\infty} \sum_{n=0}^{N} r^n\,,$$

whenever this limit exists. Our formula for the sum of a geometric series can be rewritten: $\sum\limits_{n=0}^{\infty} cr^n = c\,\frac{1}{1-r}$ for $|r| < 1$. This means:

$$\boxed{g(x) \;=\; \sum_{n=0}^{\infty} x^n \;=\; \begin{cases} \dfrac{1}{1-x} & \text{if } |x| < 1 \\ \text{undefined} & \text{if } |x| \geq 1. \end{cases}}$$

The set of $x$ for which the series converges is called the *interval of convergence*: in this case it is $-1 < x < 1$, i.e. $x \in (-1, 1)$.

Another example:

$$g(x) \;=\; \sum_{n=0}^{\infty} 3^{n-1}x^{n+1} \;=\; \sum_{n=0}^{\infty} \frac{x}{3}(3x)^n \;=\; \frac{x}{3}\,\frac{1}{1-3x} \;=\; \frac{x}{3-9x}\,,$$

provided $|3x| < 1$. The interval of convergence is thus $x \in (-\frac{1}{3}, \frac{1}{3})$.

**Testing convergence.** We usually cannot find a neat formula for the sum of an infinite series $\sum_{i=1}^{\infty} a_i = \lim_{n\to\infty} \sum_{i=1}^{n} a_i$, but we still wish to know whether the series converges to some finite value. The most obvious way to analyze this is:

> *Nth Term Non-vanishing Test:* If $\lim\limits_{n\to\infty} a_n \neq 0$, then the series $\sum_{i=1}^{\infty} a_i$ diverges. If $\lim\limits_{n\to\infty} a_n = 0$, then the series might converge or diverge.

EXAMPLES: Use the Non-Vanishing Test to detect divergence.

- $\sum_{n=0}^{\infty} 3^{n-1}x^{n+1}$ from the previous example. For the Non-vanishing Test, we want to know if the limit of the terms vanishes, $\lim\limits_{n\to\infty} 3^{n-1}x^{n+1} \stackrel{?}{=} 0$. Here $3^{n-1}x^{n+1} = (3x)^{n-1}x^2$, and $x$ is fixed while $n$ gets bigger. We see the limit is non-zero provided $|3x| \geq 1$, or $|x| \geq \frac{1}{3}$: in this case the series diverges.

  For $|x| < \frac{1}{3}$, the terms do approach zero, but in this case the Non-vanishing Test cannot determine convergence or divergence (though we know from our analysis of geometric series that it really does converge).

- $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$. Here $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{1}{n} = 0$, so the Non-vanishing Test cannot determine whether the series converges.†

†We will see in §11.3 that it diverges by the Integral Test. Or estimate by grouping terms:

$$1 + \underbrace{\tfrac{1}{2}}_{1} + \underbrace{\tfrac{1}{3} + \tfrac{1}{4}}_{2} + \underbrace{\tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8}}_{4} + \cdots > 1 + \tfrac{1}{2} + 2\cdot\tfrac{1}{4} + 4\cdot\tfrac{1}{8} + \cdots = 1 + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} + \cdots.$$