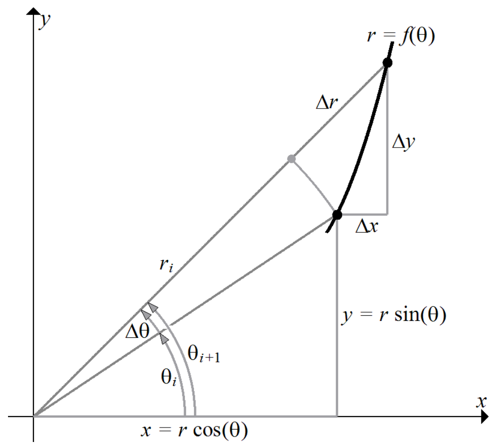
Polar Areas and Lengths

Slope in polar coordinates. We have seen that round, turny shapes are more simply described by polar $r\theta$-equations than by rectangular xy-equations. In this section, we use polar equations to compute geometric information.

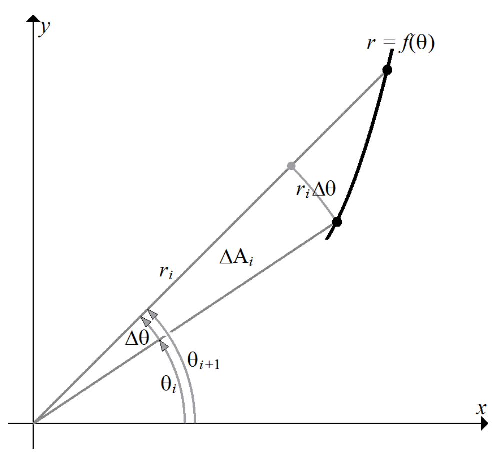
Thus, we consider a polar curve $r = f(\theta)$ over $\theta \in [a, b]$. We split the interval $\theta \in [a, b]$ into a large number n of increments, each of length $\Delta\theta = \frac{b-a}{n}$, with sample points $\theta_1, \ldots, \theta_n$. Here is a typical increment of the curve over $\theta \in [\theta_i, \theta_{i+1}]$, showing the corresponding increments in the coordinates:



Our first problem is to find the slope of this curve at a given θ. It is *not* the derivative $f'(\theta) = \frac{dr}{d\theta}$, which is the rate of change of the radius with respect to the angle. Rather, slope is the rate of change of $y = r\,\sin(\theta) = f(\theta)\,\sin(\theta)$ with respect to $x = r\,\cos(\theta) = f(\theta)\,\cos(\theta)$. That is:

$$(\text{slope at }\theta) \;=\; \frac{dy}{dx} \;=\; \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} \;=\; \frac{(f(\theta)\,\sin(\theta))'}{(f(\theta)\,\cos(\theta))'} \;=\; \frac{f'(\theta)\,\sin(\theta) + f(\theta)\,\cos(\theta)}{f'(\theta)\,\cos(\theta) - f(\theta)\,\sin(\theta)}\,.$$

Area in polar coordinates. Assume $r = f(\theta) \geq 0$ for $\theta \in [a, b]$ to avoid complications with signs, and consider the region inside the curve, defined by $0 \leq r \leq f(\theta)$ for $\theta \in [a, b]$. Apply Slice Analysis (§5.2), splitting the area A into n thin wedges ΔA_i over $[\theta_i, \theta_{i+1}]$:



We must compute the wedge area ΔA_i. Since $\Delta\theta$ is tiny, the small curve segments are very close to straight lines, and ΔA_i is a very thin triangle. Neglecting the small piece with

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radius larger that r_i, the slice ΔA_i is approximately an isosceles triangle with height r_i and base $r_i \Delta\theta$.* Thus:

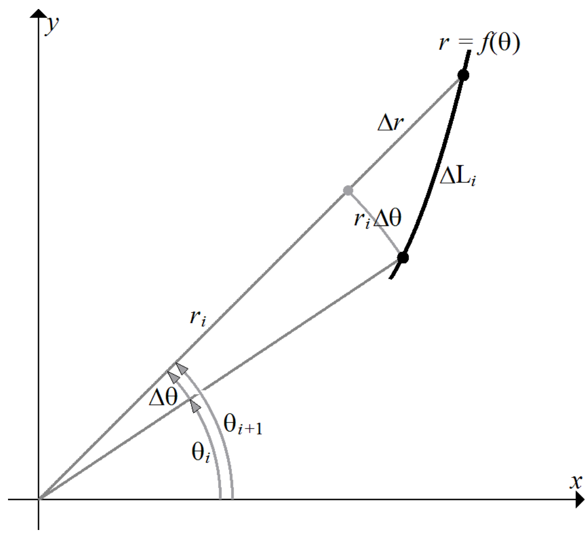
$$\Delta A_i \;\approx\; \tfrac{1}{2}(\text{base})\times(\text{height}) \;\approx\; \tfrac{1}{2}(r_i\Delta\theta)r_i \;=\; \tfrac{1}{2}r_i^2\Delta\theta\,.$$

The total area is the sum of these pieces, which is clearly a Riemann sum for an integral:

$$A \;=\; \lim_{n\to\infty}\sum_{i=1}^{n}\Delta A_i \;=\; \lim_{n\to\infty}\sum_{i=1}^{n}\tfrac{1}{2}r_i^2\Delta\theta \;=\; \lim_{n\to\infty}\sum_{i=1}^{n}\tfrac{1}{2}f(\theta_i)^2\Delta\theta \;=\; \int_a^b \tfrac{1}{2}f(\theta)^2\,d\theta\,.$$

That is, the area inside a polar graph $r = f(\theta)$ is given by an integral formula, but a different integral from the area under a rectangular graph $y = f(x)$.

Arclength in polar coordinates. Finally, we compute the length of the curve $r = f(\theta)$ for $\theta \in [a, b]$. The length L is a sum of n increments ΔL_i:



Each increment ΔL_i is approximately a straight line segment. Next to it is the radial segment Δr and the tiny circular arc with length $r_i\,\Delta\theta$, which is also approximately a straight line. We get an approximate right triangle with hypotentuse ΔL_i and legs $r_i\,\Delta\theta$ and Δr, so the Pythagorean Theorem gives:

$$\Delta L_i \;\approx\; \sqrt{(r_i\Delta\theta)^2+(\Delta r)^2} \;=\; \sqrt{\tfrac{(r_i\Delta\theta)^2+(\Delta r)^2}{(\Delta\theta)^2}}\,\Delta\theta \;=\; \sqrt{r_i^2+(\tfrac{\Delta r}{\Delta\theta})^2}\,\Delta\theta\,.$$

Therefore the total arclength is:

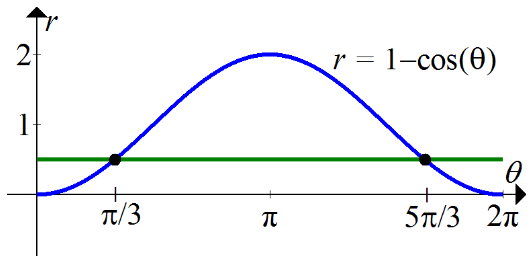
$$L \;=\; \lim_{n\to\infty}\sum_{i=1}^{n}\Delta L_i \;=\; \lim_{n\to\infty}\sum_{i=1}^{n}\sqrt{r_i^2+(\tfrac{\Delta r}{\Delta\theta})^2}\,\Delta\theta$$

$$=\; \lim_{n\to\infty}\sum_{i=1}^{n}\sqrt{f(\theta_i)^2+(\tfrac{\Delta f(\theta_i)}{\Delta\theta})^2}\,\Delta\theta \;=\; \int_a^b \sqrt{f(\theta)^2+f'(\theta)^2}\,d\theta\,.$$

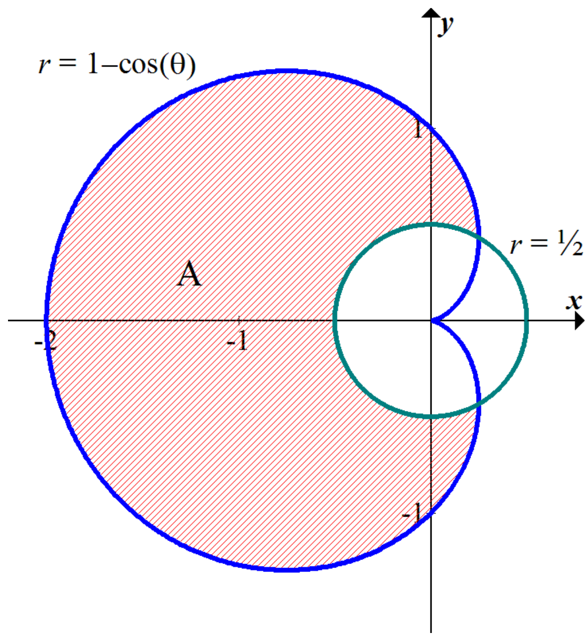
We could also deduce this from our previous parametric arclength formula (§10.3) by applying it to $(x(t), y(t)) = (f(t)\cos(t), f(t)\sin(t))$.

*On a circle of radius r, and arc of θ radians has length $r\theta$: this is the definition of radian measure.

Example: Area of intersections. Consider the polar curve $r = f(\theta) = 1 - \cos(\theta)$. We picture the abstract function f by its rectangular graph in θr parameter space (end §10.3):



The polar graph is a cardioid (heart-shape), which we draw along with the circle $r = \frac{1}{2}$.



PROB: Find the area of the crescent-shaped region inside the cardioid & outside the circle.

We must first determine the intersection points of the two curves, where:

$$r = 1 - \cos(\theta) = \tfrac{1}{2} \quad \Longrightarrow \quad \cos(\theta) = \tfrac{1}{2} \quad \Longrightarrow \quad \theta = \pm\tfrac{\pi}{3} + 2n\pi\,,$$

where n is any integer. Since the whole cardioid is traced by $\theta \in [0, 2\pi]$, we can take all intersection points in this range: $\theta = \frac{\pi}{3}$ and $\theta = -\frac{\pi}{3} + 2\pi = \frac{5\pi}{3}$. Now we take the area inside the cardioid $r = f(\theta) = 1 - \cos(\theta)$, minus the area inside the circle $r = g(\theta) = \frac{1}{2}$:

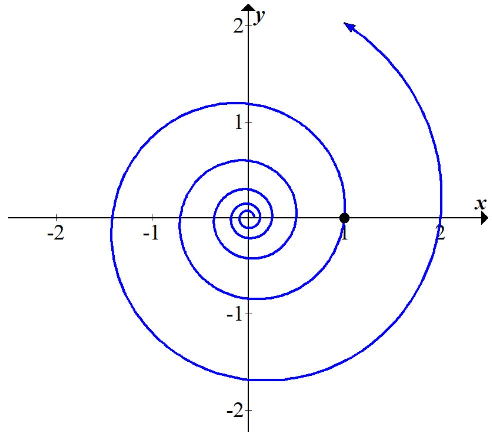
$$A \;=\; \int_a^b \tfrac{1}{2}f(\theta)^2 - \tfrac{1}{2}g(\theta)^2 \; d\theta \;=\; \int_{\pi/3}^{5\pi/3} \tfrac{1}{2}(1 - \cos(\theta))^2 - \tfrac{1}{2}(\tfrac{1}{2})^2 \; d\theta$$

$$= \; \Big[\tfrac{5}{8}\theta - \sin(\theta) + \tfrac{1}{8}\sin(2\theta)\Big]_{\theta=\pi/3}^{\theta=5\pi/3} \;=\; \tfrac{7}{8}\sqrt{3} + \tfrac{5\pi}{6} \;\approx\; 4.1\,.$$

To do the integral, expand $(1 - \cos(\theta))^2$ and use $\cos^2(\theta) = \frac{1}{2} + \frac{1}{2}\cos(2\theta)$ (see §7.2).

PROB: Find the highest point of the cardioid: $y(t) = (1 - \cos(t))\sin(t) = \max$. From §3.1, we need to solve $y'(t) = 0$. Using the Product Rule and $\sin^2 = 1 - \cos^2$ we get: $y'(t) = \sin(t)\sin(t) + (1 - \cos(t))\cos(t) = -2\cos^2(t) + \cos(t) + 1 = 0$, and the Quadratic formula gives $\cos(t) = -\frac{1}{2}$, $t = \pm\frac{2\pi}{3}$. The maximum is $y(\frac{2\pi}{3}) = \frac{3\sqrt{3}}{4}$.

Review example: Exponential spiral. Consider a snail-shell spiral curve which doubles in radius with each turn:



This is the polar graph $r = f(\theta) = ca^\theta = ce^{b\theta}$ of a general exponential function (§6.4). Assuming $f(0) = 1$, $f(2\pi) = 2$, allows us to solve for $c = 1$ and $a = 2^{1/2\pi} = e^{\ln(2)/2\pi}$ to get:

$$r \;=\; 2^{\theta/2\pi} \;=\; e^{b\theta} \quad \text{for} \quad b = \tfrac{\ln(2)}{2\pi}\,.$$

What is the length of this curve, from the point $(r, \theta) = (1, 0)$ all the way to the center, that is, for $\theta \in (-\infty, 0]$? We have $f'(\theta) = (e^{b\theta})' = be^{b\theta}$, so the arclength formula gives:

$$L \;=\; \int_{-\infty}^{0} \sqrt{f(\theta)^2 + f'(\theta)^2}\, d\theta \;=\; \int_{-\infty}^{0} \sqrt{1{+}b^2}\, e^{b\theta}\, d\theta \;=\; \Big[\tfrac{1}{b}\sqrt{1{+}b^2}\, e^{b\theta}\Big]_{\theta=-\infty}^{\theta=0}$$

$$=\; \tfrac{1}{b}\sqrt{1{+}b^2} \;-\; \lim_{N\to\infty} \tfrac{1}{b}\sqrt{1{+}b^2}\, e^{-N} \;=\; \sqrt{1{+}\tfrac{4\pi^2}{\ln^2(2)}} \;\approx\; 9.12\,.$$

Or we could use geometry to show that these infinitely many turns have finite length. Let L_1 be the length of the first turn $\theta \in [-2\pi, 0]$, and L_2 the length of the second turn, etc. The exponential spiral is *scale invariant*: each turn inward is the $\frac{1}{2}$ dilation of the previous turn, with half the length, so the total is a geomteric series $\sum_{n=1}^{\infty} cr^{n-1} = \frac{c}{1-r}$ (§11.2):

$$L \;=\; L_1 + L_2 + L_3 + L_4 + \cdots \;=\; L_1 + \frac{1}{2}L_1 + \frac{1}{2^2}L_1 + \frac{1}{2^3}L_1 + \cdots \;=\; \frac{L_1}{(1 - \frac{1}{2})} \;=\; 2L_1.$$

Harmonic Spiral. From the above, we may say that the inward spiral $r = 1/2^\theta$ has finite arclength as $\theta \to \infty$ because the geometric series $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is convergent. Let us instead model an inward spiral on the divergent harmonic series $\sum \frac{1}{n} = \infty$, namely $r = \frac{1}{\theta}$ for $\theta \geq 1$. Then this should have infinite arclength:

$$L \;=\; \textstyle\int_1^\infty \sqrt{(\frac{1}{\theta})^2 + (-\frac{1}{\theta^2})^2}\, d\theta \;=\; \int_1^\infty \frac{1}{\theta}\sqrt{1 + \frac{1}{\theta^2}}\, d\theta \;=\; \int_1^\infty \frac{\sqrt{1+\theta^2}}{\theta^2}\, d\theta.$$

Since the integrand (in the second integral) is clearly positive and decreasing, the Integral Test (§11.3) tells us that this diverges whenever the corresponding series diverges, namely :

$$\textstyle\sum_{n=1}^{\infty} \frac{1}{n}\sqrt{1 + \frac{1}{n^2}} \;\approx\; \sum_{n=1}^{\infty} \frac{1}{n} \;=\; \infty \text{ (divergent)}.$$

This can be justified by the Direct Comparison Test (§11.4), since $\frac{1}{n}\sqrt{1{+}\frac{1}{n^2}} > \frac{1}{n}$; or the Limit Comparison Test: the ratio $\frac{a_n}{b_n} = \sqrt{1{+}\frac{1}{n^2}} \to 1$, so they have the same divergence.

Alternatively, we can directly integrate, switching the variable to $\int \frac{\sqrt{1+x^2}}{x^2}\,dx$. Since we have $\sqrt{1+x^2}$ (§7.3), we try the trig substitution $x = \tan(t)$, $\sqrt{1+x^2} = \sec(t)$, $dx = \sec^2(t)\,dt$:

$$\int \frac{\sqrt{1+x^2}}{x^2}\,dx \;=\; \int \frac{\sec(t)}{\tan^2(t)}\sec^2(t)\,dt \;=\; \int \frac{1}{\sin^2(t)\cos(t)}\,dt \;=\; \int \frac{1}{\sin^2(t)\cos^2(t)}\cos(t)\,dt.$$

As in §7.2, we do the substitution $u = \sin(t)$, $1{-}u^2 = \cos^2(t)$, $du = \cos(t)\,dt$:

$$\int \frac{1}{\sin^2(t)\cos^2(t)}\cos(t)\,dt \;=\; \int \frac{1}{u^2(1{-}u^2)}\,du \;=\; \int \frac{A}{u^2}+\frac{B}{u}+\frac{C}{1{+}u}+\frac{D}{1{-}u}\,du.$$

Here the result is a rational function, expanded by partial fracitions (§7.4). Then:

$$\frac{1}{u^2(1{-}u^2)} \;=\; \frac{1{-}u^2}{u^2(1{-}u^2)}+\frac{u^2}{u^2(1{-}u^2)} \;=\; \frac{1}{u^2}+\frac{1}{(1{-}u)(1{+}u)} \;=\; \frac{1}{u^2}+\frac{C}{1{+}u}+\frac{D}{1{-}u}.$$

We can find the remaining coefficients by clearing denominators to get

$$1 = C(1{-}u) + D(1{+}u).$$

Substituting $u = 1$ gives $D = \frac{1}{2}$, and $u = -1$ gives $C = \frac{1}{2}$. The integral becomes:

$$\int \frac{1}{u^2}+\frac{\frac{1}{2}}{1{+}u}+\frac{\frac{1}{2}}{1{-}u}\,du \;=\; -\frac{1}{u}+\tfrac{1}{2}\ln(1{+}u)-\tfrac{1}{2}\ln(1{-}u) \;=\; -\frac{1}{u}+\frac{1}{2}\ln\!\left(\frac{1{+}u}{1{-}u}\right).$$

Now we need to restore the original variable $x = \tan(t)$ from $u = \sin(t)$. The standard triangle for $x = \tan(t)$ implies $u = \sin(t) = \frac{x}{\sqrt{1+x^2}}$. After simplification, the final answer is:

$$\int \frac{\sqrt{1{+}x^2}}{x^2}\,dx \;=\; -\frac{\sqrt{1{+}x^2}}{x}+\frac{1}{2}\ln\!\left(\frac{\sqrt{1{+}x^2}+x}{\sqrt{1{+}x^2}-x}\right).$$

Therefore the total arclength is:

$$L \;=\; \int_1^\infty \frac{\sqrt{1{+}x^2}}{x^2}\,dx \;=\; \lim_{x\to\infty}\tfrac{1}{2}\ln\!\left(\frac{\sqrt{1{+}x^2}+x}{\sqrt{1{+}x^2}-x}\right)+K$$

for a constant K. In the fraction $\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}-x}$, we *cannot* use L'Hopital's Rule (§6.8), since the numerator clearly approaches ∞, but the denominator does not. Substitute $x = 1/z$:

$$\lim_{x\to\infty}\sqrt{1{+}x^2}-x \;=\; \lim_{z\to 0^+}\sqrt{1{+}\frac{1}{z^2}}-\frac{1}{z} \;=\; \lim_{z\to 0^+}\frac{\sqrt{z^2{+}1}-1}{z}.$$

This is a $\frac{0}{0}$ limit, so we *can* apply L'Hopital to get:

$$\lim_{x\to\infty}\sqrt{1{+}x^2}-x \;=\; \lim_{z\to 0^+}\frac{(\sqrt{z^2{+}1}-1)'}{(z)'} \;=\; \lim_{z\to 0^+}\frac{\frac{z}{\sqrt{z^2+1}}}{1} \;=\; 0^+.$$

After all that, we obtain the expected arclength:

$$L \;=\; \tfrac{1}{2}\ln\!\left(\frac{\infty}{0^+}\right)+K \;=\; \tfrac{1}{2}\ln(\infty)+K \;=\; \infty\,.$$