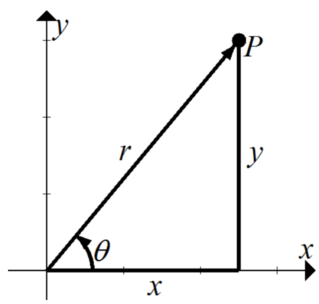
Polar Coordinates

Points in polar coordinates. The first and greatest achievement of modern mathematics was Descartes' description of geometric objects by numbers, using a *system of coordinates*. In the simplest example, *Cartesian* or *rectangular* coordinates on the plane locate a point P in terms of two coordinate measurements x and y: how far over and how far up the point is, moving parallel to the marked axes. We loosely say that P "is" the pair (x, y), because the coordinates tell how to get there from the origin. The name P is like identifying a house as "the Jones place", whereas the coordinates are like saying "the third house to the right on the second street down".

In this section, we learn how to locate the point P using a different pair of measurements, the *polar coordinates* (r, θ). The *radius* r is the distance from the origin. The *angle* θ is measured couterclockwise in radians, from the positive x-axis ray to the ray from orgin through the point P. This is like pointing to "the house 500 yards in *that* direction".



Unlike rectangular coordinates, the polar coordinates of a point are multivalent, having many equivalent versions because of the ambiguity of angles. For example, the point $(x, y) = (0, 1)$ on the positive y-axis corresponds to $(r, \theta) = (1, \frac{\pi}{2})$, where $\theta = \frac{\pi}{2}$ means a $\frac{1}{4}$ turn counterclockwise from the positive x-axis. However, we could equally well get to this point by a $\frac{3}{4}$ turn clockwise, giving $(r, \theta) = (1, -\frac{3\pi}{2})$. In fact, we could get to the point by $1\frac{1}{4}$ turns counterclockwise, $1\frac{3}{4}$ clockwise, etc. In general, we must consider all angles that differ by a multiple of a full turn 2π as the same, meaning they define the same point:

$$(r, \theta) = (r,\ \theta{+}2n\pi) \quad \text{for any integer } n.$$

It is also useful to allow negative radius: $(-r, \theta)$ means to move out along the line at angle θ, but in the opposite direction from the positive ray, along the ray $\theta \pm \pi$; thus:

$$(-r, \theta) = (r,\ \theta \pm \pi)\,.$$

There is even more ambiguity for the origin $(x, y) = (0, 0)$, which can be written as $(r, \theta) = (0, \theta)$ for any angle at all.

Both types of coordinates completely locate a point, so given either (x, y) or (r, θ),

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we can find the other by simple trigonometric formulas:

$$\text{Given } (r,\theta) \qquad \Longrightarrow \qquad \text{find } (x,y) \text{ with } \begin{cases} x &= r\cos(\theta) \\ y &= r\sin(\theta)\,. \end{cases}$$

$$\text{Given } (x,y) \qquad \Longrightarrow \qquad \text{find } (r,\theta) \text{ with } \begin{cases} r &= \sqrt{x^2+y^2} \\ \theta &= \arctan(\frac{y}{x})\,. \end{cases}$$

Here we get θ from the defining formula $\tan(\theta) = \frac{y}{x}$, and we could equally well use $\sin(\theta) = \frac{y}{\sqrt{x^2+y^2}}$, etc., always remembering we can change θ to $\theta{+}2n\pi$. Also, since $-\frac{\pi}{2} < \arctan(\theta) < \frac{\pi}{2}$, we must define $\arctan(\infty) = \frac{\pi}{2}$, $\arctan(-\infty) = -\frac{\pi}{2}$; and we must adjust the angle by $\pm\pi$ if the point lies left of the y-axis.*

Curves in polar coordinates. Any geometric object in the plane is a set (collection) of points, so we can describe it by a set of coordinate pairs. For example, the unit circle C is the set of all points at distance 1 from the origin;† the coordinates of these points form the set of all pairs (x,y) which satisfy the Pythagorean equation $x^2+y^2=1$:

$$C \;=\; \{(x,y) \text{ such that } x^2+y^2=1\}\,.$$

Again, the equality of these sets is meant loosely: a pair of numbers like $(x,y) = (\frac{3}{5},\frac{4}{5})$ is not literally a geometric point on the circle, but it *identifies* a point by means of the rectangular coordinate system. Now, polar coordinates are specially adapted to describe round, turny shapes centered at the origin, and they make the equation of the circle as simple as possible:

$$C \;=\; \{(r,\theta) \text{ such that } r=1\}\,.$$

We also have the parametric forms $(r(t),\theta(t)) = (1,t)$ and $(x(t),y(t)) = (1\cos(t), 1\sin(t))$.

EXAMPLE: The line $x+y=1$ is not at all circular or centered at the origin, and its equation becomes complicated in polar coordinates:

$$x+y=1 \quad\Longleftrightarrow\quad r\cos(\theta)+r\sin(\theta) \;=\; 1$$

$$\Longleftrightarrow \quad r \;=\; \frac{1}{\cos(\theta){+}\sin(\theta)} \;=\; \tfrac{1}{\sqrt{2}}\sec(\theta{-}\tfrac{\pi}{4})\,.$$

The last equality follows from the identity $\cos(\theta{-}\frac{\pi}{4}) = \cos(\theta)\cos(-\frac{\pi}{4}){-}\sin(\theta)\sin(-\frac{\pi}{4}) = \frac{1}{\sqrt{2}}(\cos(\theta)+\sin(\theta))$. Parametrically, $(r(t),\theta(t)) = (\frac{1}{\sqrt{2}}\sec(t{-}\frac{\pi}{4}), t)$, $t \in (-\frac{\pi}{4}, \frac{3\pi}{4})$; also $(x(t),y(t)) = \frac{1}{\sqrt{2}}\sec(t{-}\frac{\pi}{4})(\cos(t),\sin(t))$, a trajectory with constant *angular* velocity.

Similar reasoning gives the polar form of a general linear equation. For $ax+by=0$, we get $\theta = \alpha + \frac{\pi}{2}$ for the constant angle $\alpha = \arctan(\frac{b}{a})$. For constant term $c \neq 0$:

$$ax+by=c \quad\Longleftrightarrow\quad r \;=\; \frac{c}{a\cos(\theta)+b\sin(\theta)} \;=\; \frac{c}{\sqrt{a^2+b^2}}\sec(\theta{-}\alpha)\,.$$

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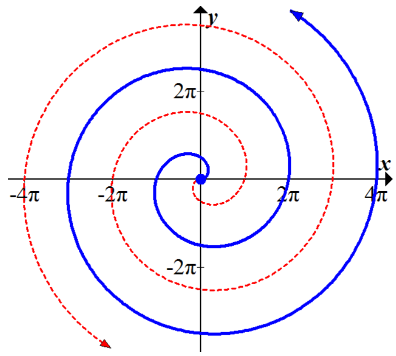
*Summarizing:

$$\theta \;=\; \arcsin\left(\frac{y}{\sqrt{x^2{+}y^2}}\right) \;=\; \arccos\left(\frac{x}{\sqrt{x^2{+}y^2}}\right) \;=\; \begin{cases} \arctan\left(\frac{y}{x}\right) & \text{if } x \geq 0 \\ \arctan\left(\frac{y}{x}\right) + \operatorname{sgn}(y)\pi & \text{if } x < 0 \end{cases}$$

The function after the last equality is called `atan2(y,x)` in computer languages.

†There is no separate curve "connecting" the points: the curve *is* just all the points.

EXAMPLE: Consider the Archimedean spiral, the shape of the groove on an old vinyl record (solid blue line).



This is defined by a point moving steadily outward as it turns around the origin: in parametric polar coordinates, $(r(t), \theta(t)) = (t, t)$ for $t \geq 0$, meaning at time t the radius and angle are both t. Converting into rectangular coordinates:

$$(x(t), y(t)) \;=\; (r(t)\cos\theta(t), r(t)\sin\theta(t)) \;=\; (t\cos(t), t\sin(t))\,, \quad t \geq 0.$$

Deparametrizing gives the $r\theta$ and xy-equations:

$$r \;=\; \theta + 2n\pi \;\text{ for integer } n \quad\Longrightarrow\quad \sqrt{x^2+y^2} \;=\; \arctan\!\left(\frac{y}{x}\right) + 2\pi n$$
$$\Longrightarrow \quad y \;=\; x\,\tan\sqrt{x^2+y^2}\,.$$

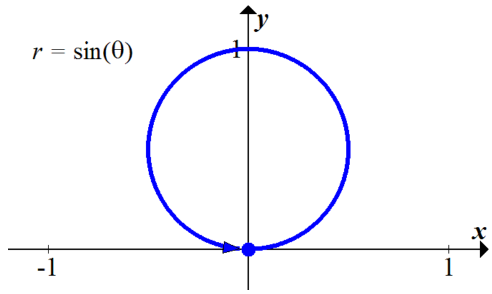
For example, we can tell the points $(x, y) = (2n\pi, 0)$ are on the spiral, because $0 = 2n\pi\tan\sqrt{(2n\pi)^2 + 0^2}$. Actually, the equation $y = x\tan\sqrt{x^2+y^2}$ defines the spiral together with its natural continuation back past its center point, namely the $\frac{1}{2}$-turn rotation of the original spiral (dashed red line).

Sketching polar graphs. Remember that a function f is just a rule taking input numbers to output numbers. It does not care what letters we use for inputs and outputs, or how we interpret those letters geometrically. We usually illustrate the function by drawing its rectangular graph $y = f(x)$, in which f controls the height y above each point on the x-axis. But another way to illustrate this function is the polar graph $r = f(\theta)$, in which f controls the radius r along each ray θ.

You can sketch the polar graph $r = f(\theta)$ by plotting points, just as for a rectangular graph. For example, consider the polar curve:

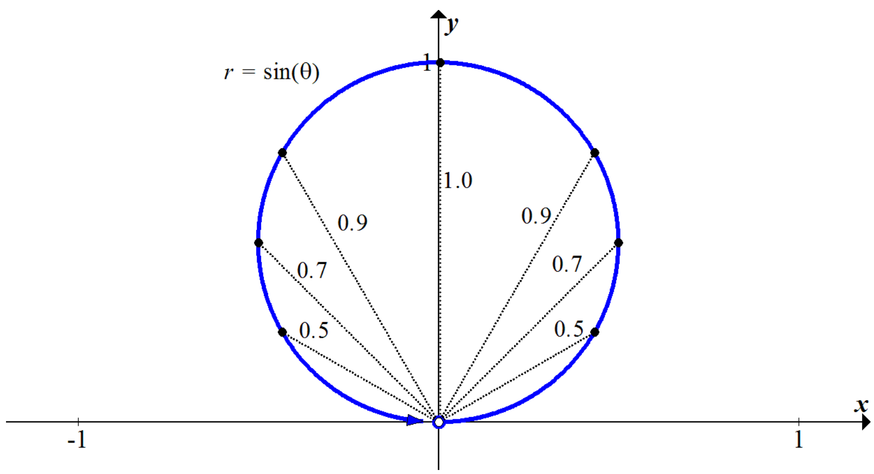
$$r = \sin(\theta)\,.$$

Imagine the plane as a field, with you standing at the origin. Look along the positive x-axis in direction $\theta = 0$, and draw a point at radius $r = \sin(0) = 0$, namely the origin itself. As you increase $\theta > 0$, turning slowly to the left, you increase the radius as $\sin(\theta)$ increases. The radius tops out at 1 when $\theta = \frac{\pi}{2}$ along the positive y-axis; and as you continue to turn, the point comes back in to the origin when $\theta = \pi$. After that, as you turn toward negative y directions the radius becomes negative, so you draw points behind you, in fact retracing the original curve.



This is a computer plot to turn the qualitative story above into an accurate graph. But you really could do this by hand, by plotting r for some standard θ:

deg	$0°$	$30°$	$45°$	$60°$	$90°$	$120°$	$135°$	$150°$	$180°$
θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
r	0	0.5	0.7	0.9	1	0.9	0.7	0.5	0



As we said, the angles $\pi \leq \theta \leq 2\pi$ give negative radius and re-plot the same points.

From the sketch, we may guess this curve is a circle, which we verify by converting to an xy-equation, and simplifying by completing the square:

$$r = \sin(\theta) \quad \Longrightarrow \quad \sqrt{x^2+y^2} = \frac{y}{\sqrt{x^2+y^2}} \quad \Longrightarrow \quad x^2+y^2-y=0$$

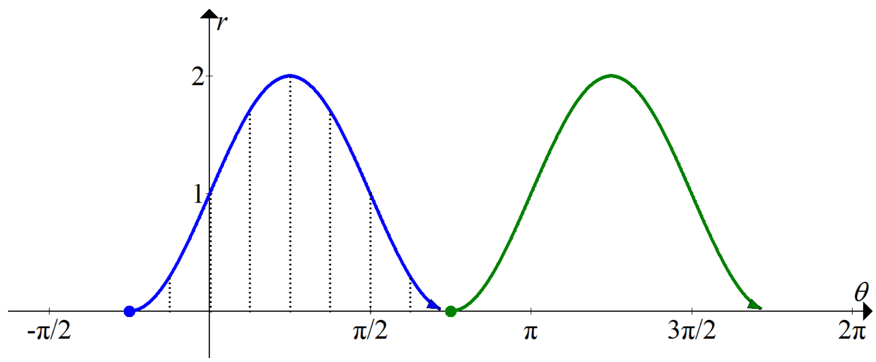
$$\Longrightarrow \quad x^2+y^2-2(\tfrac{1}{2})y+(\tfrac{1}{2})^2=(\tfrac{1}{2})^2 \quad \Longrightarrow \quad x^2+(y-\tfrac{1}{2})^2=(\tfrac{1}{2})^2\,.$$

Indeed, this is a circle of radius $\frac{1}{2}$ centered at $(x,y)=(0,\frac{1}{2})$.

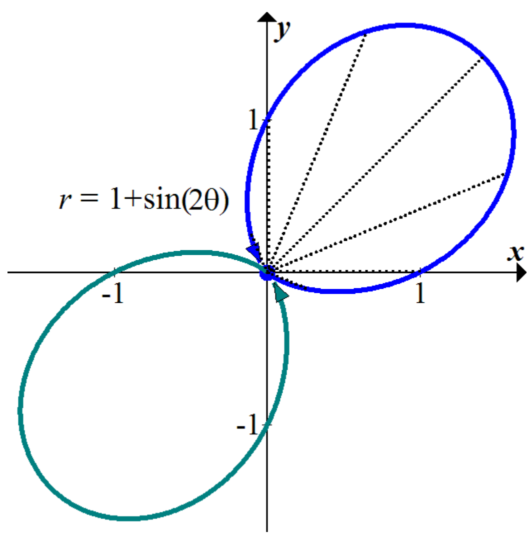
More sketching. We sketch the curve:

$$r = 1 + \sin(2\theta)\,.$$

This is more complicated, so instead of computing a table of θ and r values, we start by drawing the function $r = f(\theta) = 1 + \sin(2\theta)$ in our usual way as a rectangular graph, labeling the horizontal and vertical axes by r and θ because that is how we intend to draw them later in the polar graph.



Even without precise values, we can sketch the polar graph by adjusting the radius according to the heights of the rectangular graph (dotted lines).‡ The blue lobe is traced by $\theta \in [-\frac{\pi}{4}, \frac{3\pi}{4}]$; then the green lobe is for $\theta \in [\frac{3\pi}{4}, \frac{7\pi}{4}]$.



We have $r = 0$ at $\theta = \frac{3\pi}{4}$, so the tangent line through the origin has slope $\tan\frac{3\pi}{4} = -1$, i.e. $y = -x$. Converting the original curve to rectangular coordinates gives:

$$r \;=\; 1+\sin(2\theta) \;=\; 1+2\sin(\theta)\cos(\theta) \quad\Longrightarrow\quad \sqrt{x^2+y^2} \;=\; 1+2\frac{y}{\sqrt{x^2+y^2}}\frac{x}{\sqrt{x^2+y^2}} \;=\; \frac{(x+y)^2}{x^2+y^2}\,,$$

or $(x^2+y^2)^3 = (x+y)^4$. The polar form is much more informative about the shape!

‡Graphically, we crush the entire horizontal θ-axis in the rectangular graph to the origin in the polar graph, spreading out the radial lines like a fan.